Bireachability and final multialgebras

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Abstract. Multialgebras generalise algebraic semantics to handle nondeterminism. They model relational structures, representing relations as multivalued functions by selecting one argument as the "result". This leads to strong algebraic properties missing in the case of relational structures. However, such strong properties can be obtained only by first choosing appropriate notion of homomorphism. We summarize earlier results on the possible notions of compositional homomorphisms of multialgebras and investigate in detail one of them, the outer-tight homomorphisms which yield rich structural properties not offered by other alternatives. The outer-tight homomorphisms are different from those obtained when relations are modeled as coalgebras and the associated congruence is an inverse bisimulation equivalence. The category is cocomplete but initial objects are of little interest (essentially empty). On the other hand, the category does not, in general, possess final objects for the usual cardinality reasons. The main objective of the paper is to show that Aczel's construction of final coalgebras for set-based functors can be modified and applied to multialgebras. We therefore extend the category admitting also structures over proper classes and show the existence of final objects in this category.

1 Introduction

In the tradition of algebraic specifications, nondeterminism has been modeled by means of multialgebras, that is, algebras where operations may return not only single elements but also sets thereof, e.g., [10, 11, 13, 25, 26]. Multialgebras, or variants of power structures, have been given some attention also in the mathematical community, e.g., [19, 20, 7, 22, 4, 17], with the seminal work [14, 15] which introduced them as "algebras of complexes" to represent relational structures and demonstrated representability of Boolean algebras with operators by such algebras. [3] gives a comprehensive overview. Some variants disallow empty result-sets, e.g., [7, 24], but most do not. Then, applying the standard isomorphism

$$A_1 \times ... \times A_n \to \mathcal{P}(A) \;\simeq\; \mathcal{P}(A_1 \times ... \times A_n \times A), \tag{1.1}$$

one obtains another representation of relational structures, although with more algebraic properties, as will be observed below. This is the variant of multialgebras we will be using.

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The standard requirement put on a function $\phi : A \to B$ between two relational structures in order to obtain a homomorphism is preservation of all basic relations – for each relation symbol $R : R^A(a_1...a_n, a) \Rightarrow R^B(\phi(a_1)...\phi(a_n), \phi(a))$. This is extremely weak notion (e.g., such homomorphisms do not preserve even positive inclusions, the associated congruence is simply equivalence). Consequently, one finds in the literature numerous alternative, and stronger, requirements. In fact, the problem which we are addressing is that such proposals are *too numerous*. Of course, the choice of the notion of homomoprhism can often depend on the specific context and need not be made uniformly once and for all. But it is not certain that the possibility of such a choice is itself a virtue rather than a nuisance (especially, if we compare to the tradition of universal total algebra with the unique and powerful notion of homomorphism).

In earlier work, [23], we have shown that, restricting the possible definitions of relational/multialgebraic homomorphisms to a reasonable and almost universally followed format, there are only nine choices which are compositional. Investigation of these categories showed that only few of them are finitely complete and cocomplete. From the point of view of the semantics of (algebraic) specifications, it is desirable that the model category possesses canonical (initial or final) objects of interest. Although we have investigated only the most generic situation of the whole category of all Σ-algebras for a given signature Σ, the canonical models (when existing) were of minimal relevance (basically, empty).

This paper addresses one of the earlier investigated categories which does not, in general, possess final objects. The reason for that is the same as the reason for which the categories of coalgebras for functors involving power-set do not possess final objects – the cardinality reasons which require one to step over to the proper classes (or limit the cardinality of power-set.) We show that, making that step, we obtain final multialgebras of quite interesting nature which, in some sense, are dual to final coalgebras. The homomorphisms of multialgebraic structures in the studied category carry a similar duality to the homomorphisms induced by the coalgebraic model of (binary) relations, while the associated congruence relations are inverse bisimulations. The obtained category is cocomplete and we expect other positive results: it is, probably, complete; the homomorphisms have stronger preservation/reflection properties than the traditional (weak) ones; final objects can also be obtained for axiomatic theories. All these "probable" issues remain, however, for the future work. At the present, we only consider the existence of final objects and their character.

Section 2 gives the basic definitions, summarises earlier results and signals some possible alternatives. Section 3 presents the category of interest, "outer-tight", focussing on the notion of its congruence – bireachability. It also describes the final objects (when these exist). Section 4 generalizes this category by allowing algebras over proper classes, shows its cocompleteness and the existence of final objects. The concluding section 5 lists some open problems, suggesting also improvements and further generalizations of the obtained results. The main aspects of central constructions are summarized as proof ideas – the complete proofs will be available in a forthcoming technical report.

2 Background

Multialgebras are many-sorted algebras where operations can return (possibly empty) sets of values rather than unique values. Following [8], (one-sorted) multialgebraic operation R on a set X can be seen as a dialgebra $R : F(X) \to \mathcal{P}(X)$ in the category $\mathsf{SET}^F_{\mathcal{P}}$, where functor F gives the source of the operation and $\mathcal{P}$ is the covariant existential-image power-set functor, i.e., sending a function $\phi : A \to B$ onto $\mathcal{P}(\phi)(X) = \{\phi(x) \mid x \in X\}$, for $X \subseteq A$. The variations in the definitions of homomorphisms to be encountered below could be then seen as variations of the morphisms of dialgebras (requiring, in addition, lax transformations). Less abstractly, we use the isomorphism (1.1), and view a multialgebra as a relational structure where, for each relation, one argument is designated as its "result" and used for composition with other relations.

Definition 2.1 *For a signature $\Sigma = \langle \mathcal{S}, \mathcal{F} \rangle$, a Σ-multialgebra M is given by:*

- *a (family of) carrier set(s) $|M| = \{s^M \mid s \in \mathcal{S}\}$,*
- *a function $R^M : s_1^M \times ... \times s_n^M \to \mathcal{P}(s^M)$ for each $R : s_1 \times ... \times s_n \to s \in \mathcal{F}$, with composition defined through additive extension to sets, i.e. $R^M(X_1, ..., X_n) = \bigcup_{x_i \in X_i} R^M(x_1, ..., x_n)$.*

The only structures addressed in the paper are multialgebras, so "multialgebra" and "algebra" will be used interchangeably. We assume a given signature with R ranging over all function/relation symbols.

Selection of the "result" argument corresponds, in a sense, to turning our considerations to binary relations with the additional operation of tupling the arguments. Composition of relations $R_1 : X_{11}...X_{1n} \to X_1, ..., R_k : X_{k1}...X_{kn} \to X_k$ and $R : X_1...X_k \to X$, corresponds to application of R to the tupling $\langle R_1...R_k \rangle$. We will freely switch between relational and functional notation, so the composition can be written as $R(R_1(x_1)...R_k(x_k))$ or $\langle R_1...R_k \rangle; R$. We write composition in diagrammatic order, $R; \phi$, resp. $\phi; R$, assuming implicitly ϕ to be binary (homomorphism or, strictly speaking, a tuple $\langle \phi_1, ..., \phi_{n+1} \rangle$ of unary functions, for each relevant argument/sort i.) The composition is, as just explained, an abbreviation for the multialgebraic one, i.e.:

$$\begin{array}{rcl} \langle \langle a_1...a_n \rangle, b \rangle \in R; \phi & \iff & \exists a : \langle \langle a_1...a_n \rangle, a \rangle \in R \wedge \langle a, b \rangle \in \phi_{n+1} \\ \textit{resp. } \langle \langle a_1...a_n \rangle, b \rangle \in \phi; R & \iff & \exists b_1...b_n : \langle a_i, b_i \rangle \in \phi_i \wedge \langle \langle b_1...b_n \rangle, b \rangle \in R \end{array} \quad (2.2)$$

Having made these precautions, we will write things as if all relations were binary, algebras were one-sorted and homomorpisms simple functions (and not their families), but all considerations apply to the general case.

Selection of the "result" among the relational arguments leads to more algebraic structure reflected by homomorphisms. (In particular, derived operators of a multialgebra are analogous to those of classical algebra: for a signature Σ, the term structure T_Σ is itself a Σ-algebra, and preservation/reflection of Σ operations leads to the corresponding behaviour of the derived operators. For relational structures, derived operators are just boolean operators only very weakly related to the actual signature and not necessarily preserved by the homomorphisms preserving the basic relations. [5], V.3, p.203, considers this the reason for the subordinate role of homorphisms in the study of relational structures.) However, the study of the obtained structure is not significantly simplified. As a matter of fact, the number of possible definitions of homomorphisms, congruences, etc. does not decrease. As the first step towards simplification of the rather complicated picture, we have earlier in [23] classified compositional homomorphisms of (relational structures modeled as) multialgebras and checked finite (co)completeness of the respective categories. We recall now these results in order to motivate our choice of the outer-tight homomorphisms.

Definition 2.3 *A definition $\Delta[\_]$ of a function $\phi : |A| \to |B|$ being a homomorphism of the multialgebraic structures $A \to B$ has the form:*

$$\Delta[\phi] \iff l_1[\phi]; R^A; r_1[\phi] \quad \bowtie \quad l_2[\phi]; R^B; r_2[\phi]$$

where $l[\_]$'s and $r[\_]$'s are relational expressions (using only relational composition and inverse), and $\bowtie \in \{=, \subseteq, \supseteq\}$.

One can certainly consider other formats but most proposed definitions of homomorphisms conform to this one as, in particular, do all compositional definitions which we have ever encountered.

Definition 2.4 *A definition Δ is* compositional *iff for all $\phi : A \to B$, $\psi : B \to C$, we have $\Delta[\phi]$ & $\Delta[\psi] \Rightarrow \Delta[\phi; \psi]$, i.e.:*

$$l_1[\phi]; R^A; r_1[\phi] \bowtie l_2[\phi]; R^B; r_2[\phi] \quad \& \quad l_1[\psi]; R^B; r_1[\psi] \bowtie l_2[\psi]; R^C; r_2[\psi]$$
$$\Rightarrow \quad l_1[\phi;\psi]; R^A; r_1[\phi;\psi] \quad \bowtie \quad l_2[\phi;\psi]; R^C; r_2[\phi;\psi]$$

Theorem 2.5 ([23]) *A definition is compositional iff it is equivalent to one of:*

$$R^A; \phi \bowtie \phi; R^B \qquad \phi^-; R^A; \phi \rhd R^B \qquad \phi^-; R^A \rhd R^B; \phi^- \qquad R^A \rhd \phi; R^B; \phi^-$$

where $\bowtie \in \{=, \subseteq, \supseteq\}$ and $\rhd \in \{=, \supseteq\}$.

The following table summarises the naming conventions for the compositional cases. The name consists of two parts, the first (inner/left/...) indicating one of the four main cases in the theorem and the second (closed/tight/weak) the choice of the set relation. For the weak case there are no further distinctions, since all such cases are, in fact, equivalent. (They would not be equivalent if (homo)morphisms were relations – [6] analyses these four weak cases of "simulations", though without addressing the issue of (co)completeness.)

	$R^A; \phi \bowtie \phi; R^B$	$\phi^-; R^A; \phi \rhd R^B$	$\phi^-; R^A \rhd R^B; \phi^-$	$R^A \rhd \phi; R^B; \phi^-$
	inner	left	outer	right
closed: $\supseteq$	$\mathsf{MAlg}_{IC}(\Sigma)$	$\mathsf{MAlg}_{LC}(\Sigma)$	$\mathsf{MAlg}_{OC}(\Sigma)$	$\mathsf{MAlg}_{RC}(\Sigma)$
tight: $=$	$\mathsf{MAlg}_{IT}(\Sigma)$	$\mathsf{MAlg}_{LT}(\Sigma)$	$\mathsf{MAlg}_{OT}(\Sigma)$	$\mathsf{MAlg}_{RT}(\Sigma)$
weak: $\subseteq$	$\mathsf{MAlg}_{W}(\Sigma)$			

	initial	co-prod.	co-equal.	final	prod.	equal.
$\mathsf{MAlg}_W(\Sigma)$	+	+	+	+	+	+
$\mathsf{MAlg}_{IC}(\Sigma)$	−	−	−	+	−	−
$\mathsf{MAlg}_{IT}(\Sigma)$	−	−	+	−	−	−
$\mathsf{MAlg}_{LC}(\Sigma)$	−	−	+	+	−	−
$\mathsf{MAlg}_{LT}(\Sigma)$	−	−	+	−	−	−
$\mathsf{MAlg}_{OC}(\Sigma)$	+	+	−	+	−	+
$\mathsf{MAlg}_{OT}(\Sigma)$	+	+	+	+/−	?	+
$\mathsf{MAlg}_{RC}(\Sigma)$	+	+	+	+	+	+
$\mathsf{MAlg}_{RT}(\Sigma)$	+	−	−	−	−	+

Table 2.5. Finite limits and co-limits in the categories of multialgebras

The earlier results concerning finite (co)completeness of these categories are summarised in table 2.5.

The present paper addresses the category of outer-tight homomorphisms (the double row in the table) and, in particular, the position marked +/−. First, however, a few words about the possible alternatives.

Remark 2.6 *Viewing (binary) relations as coalgebras for the existential-image power-set functor, yields the homomorphism condition $R^A;\phi = \phi;R^B$, that is, the inner-tight homomorphisms. As we see from the table, the category $\mathsf{MAlg}_{IT}(\Sigma)$ has rather few (co)limits. This, of course, looks suspicious, since we know from [21] that any category of coalgebras over sets will be, at least, cocomplete. The difference is, however, due to the fact that although the homomorphism conditions are the same, the respective representations of relations are not.*

The absence of final objects is here due to the fact that the table addresses only categories based on sets. The non-existence of colimits is due to the algebraic character of operations, in partcular, constants which correspond to predicates. For instance, for a signature with a single sort and constant $c :\to S$, the category $\mathsf{MAlg}_{IT}(\Sigma)$ has no intial multialgebra I – for any (in particular, empty) c^I there is no IT-homomorphism $\phi : I \to A$ making $\phi(c^I) = c^A$ when $|c^I| < |c^A|$. In a coalgebra, a (predicate) constant is an arrow $c : X \to \mathbf{2}$ and this enables one to achieve commutativity, $c^A;\phi = \phi;c^B$, also when $X = \emptyset$.

In fact, the meaning of the condition is different in the two cases: for coalgebras it requires equality of two functions while for multialgebras of two sets. As an example, take the carrier $X = \{1,2\}$ and one constant c. Let, in a multilagebra M, $c^M = \{1,2\}$, while in a coalgebra C, $c(1) = c(2) = \top$. Let $X' = \{1,2,3\}$ and $c^{M'} = \{1,2,3\}$ while in a coalgebra C', $c'(1,2,3) = \top$. Although both M and C, resp., M' and C' represent the same predicates, the inclusion $i : X \to X'$ is a coalgebraic homomorphism, since indeed $c;i = i;c'$, but it is not a multialgebraic IT-homomorphism since $i(c^M) = i(\{1,2\}) = \{1,2\} \neq \{1,2,3\} = c^{M'}$.

This might be taken as a suggestion that the multialgebraic representation of relations is not the most successful one. However, using coalgebras as models of relations is by no means straightforward. For the first, one has to decide on whether to use the functor $\mathcal{P}(X^n)$ or $\mathbf{2}^{(X^n)}$ – the difference in homomorphisms will be similar to that suggested in the above remark (between equality of sets and of functions). In either case one has to decide which power-set functor to use. Any choice involves sacrificing the pleasant and well understood behavior of polynomial functors. Additional complications arise if one wants to model many-sorted relations. (Although these are hardly theoretically demanding, they *are* complications, at least of the same order as in the case of many-sorted algebras.) Multialgebraic model, on the other hand, is in agreement with the traditional notion of relation/predicate as a subset. It deals with many-argument, as well as many-sorted, relations in the uniform and elementary way. In addition, one should remark that multialgebras were introduced not merely as representations of relational structures but of Boolean algebras with operators (central, if not always recognised, in modal logics, as Kripke-frames are such algebras) and, on the other hand, as a generalisation of algebraic semantics to handle nondeterminism (most common institutions can be naturally embedded into the institution of multialgebras, with weak homomorphisms as morphisms in the model categories, [16]). The investigation of homomorphisms arises from this background and is motivated primarily by the search for the interesting canonical objects (initial or final) for algebraic specifications with nondeterminism.

Now, weak homomorphisms are those which are most commonly used. Unfortunately, this is an extremely weak notion which is also reflected in its standard name. Although the initial objects exist, they are of little interest having all predicates and relations empty. Lifting existence of initial objects to the axiomatic classes depends, of course, on the language one wants to use, and this is by no means a clarified issue. Most approaches suggest, at least, the use of inclusions, but this again leads only to empty relations in the initial objects. Furthermore, even simplest formulae are not preserved. E.g., having two constants a, b interpreted in A as $\{1\}$, resp., $\{1, 2\}$ makes $A \models a \subseteq b$. But the inclusion, which is a weak homomorphism, into B with $a^B = \{1, 3\}$ and $b^B = \{1, 2\}$ does not preserve this formula. Counterexamples can be easily found also when we restrict attention to preservation under homomorphic images. One way would be to design a specific syntax ensuring adequate restrictions of the model classes, as was done, for instance, with membership algebras, [18]. But this amounts to an application-oriented specialisation of the problem which we are not addressing here. (Similar remarks apply to the other (co)complete category $\mathsf{MAlg}_{RC}(\Sigma)$.)

The OT-homomorphisms seem to possess many desirable properties absent in other cases, especially that of weak homomorphisms. This paper characterizes final objects in the category $\mathsf{MAlg}_{OT}(\Sigma)$ and proves their existence. Now, the $+/-$ in the table 2.5 indicates that final objects can be constructed only in special cases. In general, they do not exist for the simple cardinality reasons. In the following section, we recall a series of basic facts about this category, and illustrate the character of final objects (when they exist). We also focus on the associated notion of congruence which can be seen as an inverse bisimulation equivalence. Then, we will extend the category by allowing algebras with carriers being proper classes. In this category, final objects do exist, and we show it in the way analogous to that in which the corresponding fact is proven for the categories of coalgebras for "set-based" functors in [2].

3 The category Outer-Tight

For $\Sigma = \langle \mathcal{S}, \mathcal{F} \rangle$, an OT-homomorphism, $\phi : A \to B$, is a (family of) function(s) $\phi_i : s_i^A \to s_i^B$, for each $s_i \in \mathcal{S}$, such that for every $R \in \mathcal{F}$:

$$\phi^-; R^A = R^B; \phi^-$$

in functional notation : $\forall b_1 ... b_n \in |B| : R^A(\phi_1^-(b_1)...\phi_n^-(b_n)) = \phi_{n+1}^-(R^B(b_1...b_n))$

which for constants specializes to : $\quad c^A = \phi^-(c^B)$.

This requirement is strictly stronger than that of the weak homomrphism. Since we will be dealing exclusively with OT-homomorphisms, we will not qualify the name – saying "homomorphism", we will always mean an OT-homomorphism.

The following few facts are hardly surpirsing but they are used in later results.

Fact 3.1 *An OT-homomorphism ϕ is*

1) mono iff it is injective;
2) epi iff it is surjective;
3) iso iff it is bijective.

The following observation will not be refered to later on, but it is used in a couple of proofs of the results mentioned in the sequel.

Given $A, A' \in \mathsf{MAlg}_{OT}(\Sigma)$, A' is a subalgebra of A, $A' \sqsubseteq A$, iff the inclusion $|A'| \subset |A|$ is a homomorphism. (The categorical definition would not introduce any significant changes.) In general, an inclusion need not be a homomorphism. But the following fact holds.

Fact 3.2 *Inclusions between subalgebras of the same algebra are OT-homomorphisms. I.e., if $A_1 \sqsubseteq A$ and $A_2 \sqsubseteq A$ and $|A_2| \subset |A_1|$, then also $A_2 \sqsubseteq A_1$.*

The following fact ensures that the diagram of subalgebras is directed.

Fact 3.3 *For an algebra A and every set $X \subseteq |A|$, there is a smallest subalgebra $A_X \sqsubseteq A$ with $X \subseteq |A_X|$.*

Thus, if $A_1, A_2 \sqsubseteq A$, then there is also (a smallest) $A_3 \sqsubseteq A$, with $|A_1| \cup |A_2| \subseteq |A_3|$. In the proof, one extends appropriately the set X or, like in the classical case, verifies that intersection of subalgebras is a subalgebra.

3.1 Bireachability

In order for the quotient construction performed on a carrier of a (classical) Σ-algebra to yield a (quotient) Σ-algebra, the equivalence must be a Σ-congruence. However, for any (classical) algebra A and any equivalence $\sim$ on its carrier, the quotient $A/_\sim$, with operations collecting the possibly non-congruent results (i.e., defined by $R^{A/\sim}([a]) = \{[n] : n \in R^A(a'),\ a' \in [a]\}$), is a multialgebra, and the construction works in the same way if we start with a multialgebra, and not only a classical algebra. Defining the mapping $q : A \to A/_\sim$ by $q(a) = [a]$, the operations are obtained as $R^{A/\sim} = q^-; R^A; q$. In general, this mapping is only a weak homomorphism, just like the kernel of a weak homomorphism is, in general, only an equivalence. (This correspondence is perhaps the clearest expression of the weakness of this homomorphism notion.) OT-homomorphisms come along with a much stronger notion of a congruence.

Definition 3.4 *An equivalence $\sim$ on A is OT-congruence iff:* $\sim; R^A; \sim\ =\ \sim; R^A$

More explicitly, the inclusion $\subseteq$ says that $\forall a'', a', b, b' : a'' \sim a' R^A b' \sim b \Rightarrow \exists a \sim a'' : a R^A b$ which, when $\sim$ is equivalence, is the same as:

$$\forall a', b, b' : a' R^A b' \sim b \Rightarrow \exists a \sim a' : a R^A b. \tag{3.5}$$

Any equivalence satisfying this last condition is OT-congruence, since the opposite inclusion $\sim; R^A; \sim\ \supseteq\ \sim; R^A$ holds trivially for any reflexive $\sim$.

This characterisation of OT-congruence can be visualized as an "inverse" (bi)simulation. (Bi)simulation requires propagation of $\sim$ forward, while OT-congruence backward – we should be therefore allowed to call this relation "bireachability".[1] On the drawing, the dotted lines indicate the required existence implied by the regular lines:

$$\begin{array}{cc}
\text{(bi)simulation} & \text{bireachability} \\
\begin{tikzcd} b \arrow[r, dotted, no head, "\sim"] & b' \\ a \arrow[u, "R"] \arrow[r, no head, "\sim"] & a' \arrow[u, dotted, "R"'] \end{tikzcd}
&
\begin{tikzcd} b \arrow[r, no head, "\sim"] & b' \\ a \arrow[u, "R"] \arrow[r, dotted, no head, "\sim"] & a' \arrow[u, dotted, "R"'] \end{tikzcd} \\
\forall a, b, a' : aRb \ \&\ a \sim a' & \forall a, b, b' : aRb \ \&\ b \sim b' \\
\Rightarrow \exists b' \sim b : a'Rb' & \Rightarrow \exists a' \sim a : a'Rb'
\end{array} \tag{3.6}$$

Henceforth, we will use the words "bireachability" and "OT-congruence" as synonyms. The same meaning will be attached also to "congruence", unless the word is qualified in some other way.

Fact 3.7 *If $\phi : A \to B$ is OT then so is its kernel $\sim_\phi$.*

[1] We are not addressing any details concerning bisimulations. For the sake of analogy, since OT-congruences are equivalences, it is most convenient to think of bisimulation defined as a symmetric simulation, rather than merely as a simulation with inverse being also a simulation. Exact duality obtains between our bireachability and the equivalences satisfying the condition that for every $R : \sim; R^A; \sim\ =\ R^A; \sim$, i.e., IT-congruences or bisimulations in (3.6), referred to in remark 2.6. In [4] such equivalences were called "preserving the arguments" (as opposed to congruences "preserving the valuess"). In [9], the relation dual to mere simulation, without the requirement of equivalence, was called "opsimulation," but the name "biopsimulation" does not seem very appealling.

The inverse does not hold generally; even if the kernel of ϕ is OT, ϕ itself may be not. We have a slightly weaker claim.

Fact 3.8 *If $\sim$ is a bireachability then the mapping $q : A \to A/_\sim$, $q(a) = [a]$, is an OT-epimorphism.*

This allows us to obtain epi-mono factorisation of morphisms in $\mathsf{MAlg}_{OT}(\Sigma)$.

Fact 3.9 *For every homomorphism $h : A \to B$ there is a (regular) epi $e : A \to Q$ and mono $m : Q \to B$ such that $h = e; m$.*

Bireachability on a Σ-multialgebra has itself a multialgebraic Σ-structure.

Definition 3.10 *Given a bireachability $\sim$ on an $A \in \mathsf{MAlg}_{OT}(\Sigma)$, we define $A^\sim \in \mathsf{MAlg}_{OT}(\Sigma)$:*

- $|A^\sim| = \{\langle a_1, a_2\rangle : a_1, a_2 \in |A| \wedge a_1 \sim a_2\}$, *and*
- $f^{A^\sim}(\langle a_1, b_1\rangle ... \langle a_n, b_n\rangle) = \{\langle x, y\rangle : x \in f^A(a_1 ... a_n) \wedge y \in f^A(b_1 ... b_n) \wedge x \sim y\}$, *which yields*
- *for constants* $c^{A^\sim} = \{\langle x, y\rangle : x, y \in c^A \wedge x \sim y\}$.

Fact 3.11 *Given a bireachability $\sim$ on A. 1) The two projections $\pi_1, \pi_2 : A^\sim \to A$, $\pi_i(\langle a_1, a_2\rangle) = a_i$ are OT-homomorphisms. 2) Moreover, $A/_\sim$ with the quotient homomorphism $q : A \to A/_\sim$ is their coequalizer.*

Maximal bireachability. Given a collection $C = \{\sim_i : i \in I\}$ of equivalences (on a set/algebra A), one defines their lub as the transitive closure of their union, i.e., $\sim\ =\ \bigvee_i \sim_i\ =\ (\bigcup_i \sim_i)^*$. Explicitly, $a \sim a'$ iff there exists a finite sequence $a = a_0 a_1 ... a_n = a'$ and a respective sequence of the equivalences from C, $\sim_1 \sim_2 ... \sim_n$, such that $a_i \sim_{i+1} a_{i+1}$ for all $0 \leq i < n$.

The same construction applies also to bireachabilities. The following lemma will be of crucial importance.

Lemma 3.12 *Given a collection $C = \{\sim_i : i \in I\}$ of bireachabilities on a multialgebra A, then $\sim\ = \bigvee_i \sim_i$ is a bireachability.*

Notice that the maximal bireachability need not be the standard unit relation. For instance, for the algebra $\begin{array}{cc} b_1 & b_2 \\ {\scriptstyle R}\uparrow & \\ a_1 & \end{array}$, the elements b_1 and b_2 cannot be related by any bireachability, according to the observation (3.5).

One verifies easily that the construction yields, in fact, the least upper bound – with respect to the subset relation – of the argument bireachabilities. Thus, the collection of all congruences on a multialgebra is a complete upper semilattice with the least element being identity, and so it is a complete lattice. (Greatest lower bounds are not, however, obtained as mere intersections.)

Fact 3.13 *Let $B \sqsubseteq A$, $\sim_A$ be a bireachability on A, and $\sim_B \subseteq \sim_A$ be restriction of $\sim_A$ to the carrier of B, i.e., $\sim_A \cap |B| \times |B|$. Then $\sim_B$ is bireachability on B.*

3.2 Final objects in $\mathsf{MAlg}_{OT}(\Sigma)$

Final objects do not exist in $\mathsf{MAlg}_{OT}(\Sigma)$ due to the usual cardinality reasons. (A multialgebra for an operation $f : S \to S$ is essentially a coalgebra for the existential-image power-set functor.) As stated in the introduction $\mathsf{MAlg}_{OT}(\Sigma)$ is finitely cocomplete but the existence of final objects has been shown only for a very special case. We show here such a case mainly to illustrate the interesting features of the final objects.

Example 3.14 *Let $\Sigma = \langle \{s_1, s_2\}, \{c :\to s_1; f : s_1 \to s_2\}\rangle$. The final object Z in $\mathsf{MAlg}_{OT}(\Sigma)$ can be described as follows. (Expressions like "$\emptyset_1$" or "$fc\emptyset$" are simple names – mnemonic devices – not any sets or function applications.)*

- $s_1^Z = \{c, \emptyset_1\}$, $s_2^Z = \{fc, f\emptyset, fc\emptyset, \emptyset_2\}$
- $c^Z = \{c\}$ *and* $f^Z(c) = \{fc, fc\emptyset\}$, $f^Z(\emptyset_1) = \{f\emptyset, fc\emptyset\}$.

In words, each sort contains only elements needed to distinguish any combination of operations returning the elements of this sort. In s_1^Z it is enough with one element to interpret the constant. In addition, there is always a "junky" element not belonging to the result of any operation, $\emptyset_1$. s_2^Z contains one such element, $\emptyset_2$, as well as one element characteristic for (belonging only to) $f^Z(c) \ni fc$, one for $f^Z(\emptyset_1) \ni f\emptyset$ and one for $f^Z(c) \cap f^Z(\emptyset_1) \ni fc\emptyset$.

If we had two constants of sort s_1, we would obtain corresponding collection $\{c, d, cd, \emptyset_1\}$ in s_1^Z, while s_2^Z would now contain characteristic element for every possible $f^Z(x)$ when $x \in s_1^Z$, as well as for every intersection $\bigcap_{x \in X} f^Z(x)$ for every possible $X \subseteq s_1^Z$.

Viewing the set of results of any application, $f^Z(x)$, as the set of possible (or nondeterministic) observations of its argument x, the construction amounts to providing the minimal number of elements needed for every set of (every series of) observations to have its unique characteristic result.

The most general form of this construction can be obtained when signature does not contain any "loops". Call a signature "acyclic" if there is no derived operator t with target sort occurring also among the argument sorts.

Fact 3.15 *If Σ is acyclic then $\mathsf{MAlg}_{OT}(\Sigma)$ has final objects.*

We will now extend the category $\mathsf{MAlg}_{OT}(\Sigma)$ to allow for the existence of final objects without any restrictions on the signature. As in the case of coalgebras, we have to either impose some cardinality limits or else leave the set-based categories and allow algebras with proper classes as carriers. The former case leads to rather special conditions[2] and so we follow the later alternative.

[2] E.g., final objects can be obtained if algebras considered are such that every element of the carrier can be reached from at most finite number of other elements in at most finite number of ways – the "reachability" restriction which is, in a sense, dual to restricting the $\mathcal{P}$ functor to $\mathcal{P}^{fin}$ returning only finite sets.

4 The category Outer-Tight with classes

Given a Σ with sort symbols $\{s_1...s_n\}$, we allow algebras where carrier of each sort is a class. Likewise, operations and constants can return proper classes.[3] But we will need the assumption that

$$\text{each such algebra is a colimit of its small subalgebras and, moreover, the category contains all algebras which are such colimits.} \tag{4.1}$$

Since colimit arrows are jointly epi (and, by fact 3.1, our epis are surjective) and the diagram of (small) subalgebras is directed (fact 3.3), the above assumption implies that:

$$\text{for every algebra } A \text{ and set } X \subseteq |A|\text{, there is a small subalgebra } sA \sqsubseteq A \text{ with } X \subseteq |sA|. \tag{4.2}$$

We denote this category $\mathsf{MAlg}^*_{OT}(\Sigma)$. (We will comment on more specific conditions which could replace (4.1) ensuring that all our constructions yield appropriate results in the concluding section.)

A bireachability R on an A which is a colimit of its small subalgebras A_i, is itself a colimit of its small subalgebras $R_i = R \cap |A_i| \times |A_i|$, i.e., $R \in \mathsf{MAlg}^*_{OT}(\Sigma)$. Lemma 3.12 applies unchanged when the collection is a proper class of small bireachabilities. Performing the same standard construction on the collection of *all small* bireachabilities on a given multialgebra yields the following lemma.

Lemma 4.3 *$\forall A \in \mathsf{MAlg}^*_{OT}(\Sigma)$ there exists a unique maximal bireachability $\sim_A$.*

The following easy technicality will be needed in the proof of the next lemma. (Notation follows the diagram below.)

Fact 4.4 *Let $\{A_i : i \in I\}$ be the class of small subalgebras of A (A being their colimit), R be a congruence on A and R_i the respective restrictions of R to A_i. Then the family of inclusions $\{r_i : R_i \hookrightarrow R : i \in I\}$ is jointly epi and, for every $c : A \to C$, if $\forall i \in I : \pi_{i1}; a_i; c = \pi_{i2}; a_i; c$ then $\pi_1; c = \pi_2; c$.*

The result which we will actually need is the following one.

Lemma 4.5 *Given an algebra $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$ and a congruence R on A, the quotient $A/_R$ is a colimit of its small subalgebras.*

[3] This might cause some foundational worries since functions returning classes, and hence also indexed families of classes, are not legal objects in the most common class theory, NBG. This signals that we must use an alternative foundation, Grothendieck's hierarchy of universes being the natural candidate. We will use the words "small"/"set" and "large"/"class" in the sense of being a memember of the lowest level $\mathcal{U}_1$ versus of any higher level $\mathcal{U}_i \setminus \mathcal{U}_1$ (for $i \geq 2$), respectively.

PROOF: We consider the following (schema of the) diagram:

$$\begin{tikzcd}
\mathbf{R} & R_j \arrow[r, hook, dotted, "r_{ji}"] \arrow[d, shift right, "\pi_{j2}"'] \arrow[d, shift left, "\pi_{j1}"] & R_i \arrow[r, hook, "r_i"] \arrow[d, shift right, "\pi_{i2}"'] \arrow[d, shift left, "\pi_{i1}"] & R \arrow[d, shift right, "\pi_2"'] \arrow[d, shift left, "\pi_1"] & \\
\mathbf{A} & A_j \arrow[r, hook, dotted, "a_{ji}"] \arrow[d, "q_j"] & A_i \arrow[r, hook, "a_i"] \arrow[d, "q_i"] & A \arrow[d, "q"] \arrow[ddr, dotted, "ax"] & \\
\mathbf{A}/_{\mathbf{R}} & A_j/_{R_j} \arrow[r, hook, dotted, "ar_{ji}"] \arrow[drrr, "x_j"'] & A_i/_{R_i} \arrow[r, hook, "ar_i"] \arrow[drr, "x_i"] & A/_R \arrow[dr, dotted, "x"] & \\
 & & & & X
\end{tikzcd}$$

$\mathbf{A}$, resp. $\mathbf{R}$, stand for the whole diagrams consisting of the respective small subalgebras A_i of A and $R_i = R \cap |A_i| \times |A_i|$ (by Fact 3.13, $R_i \sqsubseteq R$) with the inclusion arrows a_{ji}, resp. r_{ji}. A with inclusions a_i is colimit of $\mathbf{A}$. The collection of all r_i's, resp., all a_i's is jointly epi. All q_i's are epi.

The diagram $\mathbf{A}/_{\mathbf{R}}$ contains all quotient algebras $A_i/_{R_i}$ and inclusion arrows between them. Since for each $i : R_i = R \cap |A_i| \times |A_i|$, we have an inclusion $a_{ji} : A_j \hookrightarrow A_i$ iff $r_{ji} : R_j \hookrightarrow R_i$. But then, this implies the existence of a mono $ar_{ji} : A_j/_{R_j} \hookrightarrow A_i/_{R_i}$. For each $A_i/_{R_i}$, we can obtain an isomorphic algebra by replacing every element $[a]^{R_i}$ by $[a]^R$ (though $[a]^{R_i} \subseteq [a]^R$ and the inclusion can be proper, whenever $R(a_1, a_2)$ and $a_1, a_2 \in |A_i|$, then also $R_i(a_1, a_2)$). This – making all monos ar_i and ar_{ij} into inclusions – simplifies the argument below.

We want to show that $A/_R$ with all inclusions ar_i is colimit of $\mathbf{A}/_{\mathbf{R}}$. Obviosuly, for each (existing) ar_{ji} we do have that $ar_j = ar_{ji}; ar_i$, since all arrows are inclusions. So assume an X with arrows $x_i : A_i/_{R_i} \to X$ such that $x_j = ar_{ji}; x_i$ for all (relevant) i, j.

1. Since $q_j; ar_{ji} = a_{ji}; q_j$, we obtain that for all (relevant) $j, i : x_j = ar_{ji}; x_i \Rightarrow q_j; x_j = q_j; ar_{ji}; x_i = a_{ji}; q_i; x_i$. That is, X with $q_i; x_i$ is a commutative cocone over $\mathbf{A}$. Since A is colimit of $\mathbf{A}$, we obtain a unique arrow $ax : A \to X$ such that for all $i : q_i; x_i = a_i; ax$.

2. For every i, since $\pi_{i1}; q_i = \pi_{i2}; q_i$, so also $\pi_{i1}; q_i; x_i = \pi_{i2}; q_i; x_i$ and by 1, $\pi_{i1}; a_i; ax = \pi_{i2}; a_i; ax$. By Fact 4.4, we thus have $\pi_1; ax = \pi_2; ax$.

3. By Fact 3.11, $(A/_R, q)$ is coequalizer of π_1, π_2, and thus we obtain a unique arrow $x : A/_R \to X$ making $q; x = ax$. This is the arrow we are looking for:

4. Commutativity: $q_i; ar_i; x = a_i; q; x \overset{3.}{=} a_i; ax \overset{1.}{=} q_i; x_i$. But q_i is epi and so $ar_i; x = x_i$.

5. Uniqueness: assume another arrow $y : A/_R \to X$ with $ar_i; y = x_i$ for all i. Then also, $q_i; x_i = q_i; ar_i; y = a_i; q; y$ and thus, for every $i : a_i; q; y = a_i; q; x$. Since a_i are jointly epi, this means that $q; y = q; x$ and now, since q is epi, $x = y$. $\square$

4.1 Cocompleteness and final objects of $\mathsf{MAlg}^*_{OT}(\Sigma)$

The positive results for the category $\mathsf{MAlg}_{OT}(\Sigma)$ from table 2.5, generalise to the extended category $\mathsf{MAlg}^*_{OT}(\Sigma)$. We only mention the results needed in the construction of final objects, suggesting the constructions used in the proofs.[4]

Proposition 4.6 $\mathsf{MAlg}^*_{OT}(\Sigma)$ *has initial objects and coproducts.*

PROOF IDEA: Empty algebra is trivially an initial object.
Consider a class $\{A_i : i \in I\}$ of algebras. Its coproduct is the algebra C whose carrier is the disjoint union of the carriers of all A_i, with operations defined as:

$$f^C(\overline{x}) = \begin{cases} f^{A_i}(\overline{x}) & \text{if for all } x \in \overline{x} : x \in |A_i| \\ \emptyset & \text{otherwise} \end{cases}$$

and constants as disjoint unions: $c^C = \biguplus_i c^{A_i}$. The injections $\iota_i : A_i \to C$ are obviously OT, and C is colimit of small subalgebras (of all A_i's). $\square$

Proposition 4.7 $\mathsf{MAlg}^*_{OT}(\Sigma)$ *has all coequalizers.*

PROOF IDEA: Given two arrows $\phi_1, \phi_2 : A \to B$, we start as usual by considering the equivalence closure $\sim$ on B of the relation $E = \{\langle \phi_1(a), \phi_2(a) \rangle : a \in |A|\}$. Equivalence classes induced by this relation are denoted $B_1, B_2 \ldots$. Assuming the global axiom of choice, we can choose the representatives $b_i \in B_i$, and the carrier of the coequalizer object C is the collection of such representatives. (We may occasionally write $[b_i]$ for B_i.)[5] Operations are defined by:

$$b_2 \in f^C(b_1) \iff B_2 \subseteq f^B(B_1)$$

which for constants specializes to: $b_i \in c^C \iff B_i \subseteq c^B$. The arrow $ce : B \to C$ is the usual $\forall x \in B_i : ce(x) = b_i$. By the definition of $\sim$, it makes $\phi_1; ce = \phi_2; ce$. Verification that it is OT and of the universality is rather lengthy and technical. $\square$

We have thus shown that the assumption (4.1), according to which $\mathsf{MAlg}^*_{OT}(\Sigma)$ contains only those colimits of small algebras which happen to exist there, indeed is a category with *all* colimits. The main result is now obtained from the following lemma (with a straightforward proof).

Lemma 4.8 *For a given multialgebra A, let $\sim_A$ denote the maximal congruence on A (existing by Lemma 4.3). For any algebra B there is at most one homomorphism $B \to A/_{\sim_A}$.*

[4] Notice that due to the difference in the definition of homomorphism, cocompleteness of $\mathsf{MAlg}_{OT}(\Sigma)$ is not a special case of the general fact about dialgebras, according to which the category SET^F_G has all colimits preserved by the functor F.

[5] In case some of the equivalence classes B_i's are proper classes, we have to follow the trick of Dana Scott (quoted in [1], Appendix B) in order to obtain the quotient, i.e., to consider as B_i only its subset of the elements having the least possible rank in the cummulative hierarchy.

Theorem 4.9 $\mathsf{MAlg}^*_{OT}(\Sigma)$ *has final objects.*

PROOF: Let C be a coproduct of all small algebras in $\mathsf{MAlg}^*_{OT}(\Sigma)$ (which exists by proposition 4.6). For every $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$ there exists (at least one) arrow $A \to C$ since, by (4.1), A is a colimit of its small subalgebras, and there is an arrow from each such to C.

Let $\sim_C$ be the maximal congruence on C (existing by lemma 4.3), and let $Z = C/_{\sim_C}$. We thus obtain (at least) one arrow from every algebra to Z and, by lemma 4.8, this arrow is unique. By lemma 4.5, Z is colimit of its small subalgebras and hence belongs to $\mathsf{MAlg}^*_{OT}(\Sigma)$. $\square$

5 Conclusion

Multialgebras lie at the intersection of several research currents. They

- represent relations and, generally, Boolean algebras with operators;
- generalise traditional algebras, in particular,
- provide a fundamental instance of power structure construction;
- with one-argument operations, provide particular examples of coalgebras;
- provide a specific and well-motivated example of dialgebras, [8].

The apparently poor algebraic structure and, on the other hand, a multiplicity of choices when generalising most of the standard notions might discourage investigation of multialgebras. We have argued that, as far as the notion of homomorphism is concerned, the number of choices is, after all, not so large and in fact limited to one, while further choices are mainly conditioned by this one. The category of multialgebras with outer-tight homomorphisms is cocomplete and the associated notion of congruence – bireachability – arises as an inverse to the bisimulation equivalence.

We have shown that the category $\mathsf{MAlg}^*_{OT}(\Sigma)$ of multialgebras (admitting proper classes as carriers) possesses final objects with interesting structure which reflects the reachability relation in the way analogous to final coalgebras reflecting the similarity relation. We have considered only the class of all Σ-multialgebras and although we expect the existence of final objects can be lifted to (some) axiomatic classes, the possibility and scope of this lifting remain to be investigated. The question which still remains open is the existence of products which, intuitively, should be related (or even equal) to largest bireachability between the arguments. Attempts to construct counter-examples have failed and we are convinced that products do exist in $\mathsf{MAlg}^*_{OT}(\Sigma)$, but the claim and an explicit construction remain to be demonstrated. There remains also the open question concerning the more specific conditions, than those given in (4.1), on the actual algebras to be included in the category $\mathsf{MAlg}^*_{OT}(\Sigma)$. As can be seen from the proof of lemma 4.6, we must allow constants (unary predicates) to denote proper classes. We suspect that the following condition may be sufficient to ensure the existence of final objects and (co)completeness of the category: for every operation f in an algebra A : if $f^A(X)$ is a set then so is X. Sufficiency of this condition or, possibly, alternative fomulations remain to be investigated.

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