On Specialization of Derivations in Axiomatic Equality Theories

A. Pliuškevičienė, R. Pliuškevičius

Institute of Mathematics and Informatics
Akademijos 4, Vilnius 2600, LITHUANIA
email: logica@sedcs.mii2.lt

M. Walicki, S. Meldal

University of Bergen
Department of Informatics
HiB, N-5020 Bergen, NORWAY
e-mail: {michal,sigurd}@ii.uib.no

Abstract

Walicki and Meldal have defined a calculus DEQ ("Disjunctive EQuational calculus") for reasoning about nondeterministic operators when specifying nondeterministic systems in an equation-oriented style.
A variant of DEQ, the calculus DEQ^* for axiomatic equality theories with cut-like rules introducing as cut formulas only the negative equalities of a specific axiom, is constructed. For pure positive specific axioms (i.e. with empty antecedents) and for so called non-contrary equality theories DEQ^* does not contain cut-like rules at all. The variant of the calculus DEQ^* without structural rules of contraction and exchange is constructed. A simple cut-elimination procedure for axiomatic equality theories is presented.

1. Introduction

The motivation for introducing DEQ was the need for the disjunctive formulae when specifying and reasoning about nondeterministic operations [9]. The notion of nondeterminism arises naturally in describing concurrent systems. Nondeterminism is also a natural concept in describing sequential programs, either as a means of indicating a *don't care* attitude as to which among a number of computational paths will actually be utilized in a particular computation or as a means of increasing the level of abstraction [3]. In terms of satisfiability, allowing nondeterminism is quite distinct from underspecification. The latter, though admitting distinct models, still insists on a function always returning a particular value, given a particular list of arguments. In the case of nondeterministic operators this is no longer true.

In [9, 10] we presented a function oriented view of nondeterministic operators. The result of a (non-)deterministic operator is considered a single (rather than set) value, namely the result of a *particular application*. Distinct occurrences of a particular term each denote a single value, but not necessarily the same one for each. For instance, specifying an operator $\sqcup.s$ which chooses nondeterministically an element from the set s, one needs to say that for any application i, $\sqcup_i.\{0,1\} = 0 or \sqcup_i .\{0,1\} = 1$. In [9,10] it is shown how viewing nondeterministic operations as determinisitc functions dependent on an additional index (application) argument, allows one to translate nondeterministic specifications into deterministic ones using the language of conditional equations extended with disjunction. Then we presented a natural deduction system, DEQ, as a reasoning tool for transformed algebraic specifications of nondeterministic operators, using axiomatic equality theories. Along with the natural aspect and modularity, the system DEQ contains a cut rule which is not amenable to automated reasoning. This paper introduces an equivalent variant DEQ^* of the DEQ calculus where cut is replaced by cut-like rules making DEQ^* more amenable to automation.

A general theory of specialization of derivations in axiomatic equality theories was proposed by M. Rogava in [8] and A. Pliuškevičienė in [4, 5]. In [8] specific axioms of axiomatic equality theories were divided into two types: The axioms having a single equality, i.e. only one item-equality, were identified as being of *type 1*. The other specific axioms were identified as axioms of *type 2*. For specific axioms of type 1 equality-like rules were introduced in [8]. For specific axioms of type 2 unspecialized cut-like rules were introduced in [8]. In [4, 5] we introduced specialized cut-like rules for specific axioms of type 2, introducing positive equalities of specific axioms as cut formulas.

There we also introduced the complex notion of variants of equality formulas by specific axioms with a view to eliminate the introduction of the negative equalities of specific axioms as cut formulas. The main purpose of this paper is to construct the calculus DEQ^* which is equivalent to DEQ. DEQ^* contains a cut-like rule introducing as cut formulas only the negative equalities of specific axioms. Therefore in DEQ^* the complex notion of variant of [4, 5] is not necessary. For pure positive specific axioms (i.e. with empty antecedents) and for so-called non-contrary equality theories one can construct the calculus DEQ^* not containing cut-like rules at all. In this process no variant notion is used. The alternate version of the calculus DEQ^* not containing structural rules of contraction and exchange is also constructed. In the paper a simple cut-elimination procedure for axiomatic equality theories is presented.

2. The natural deduction-like calculi DEQ and DEQ'

Definition 2.1 (sequent) A sequent is an expression $\Gamma \to \Delta$, where Γ, Δ are finite sets of equalities $r = s$, where r, s are arbitrary terms. $\bigtriangledown$
Definition 2.2 (calculus DEQ). A calculus DEQ is defined by the following postulates (see e.g. [9, 10]):

(1) specific axioms (specifications)

(Sp. AX_k) $r_1 = s_1, \ldots, r_m = s_m \to u_1 = v_1, \ldots, u_n = v_n$,

where k, m, n are some natural numbers; the index k identifies the specific axioms;

(2) equality postulates (where e represents a single equation, t, t_1, t_2 are arbitrary terms)

R1. $$\to t = t$$

R2. $$\frac{\Gamma^x_{t_1} \to \Delta^x_{t_1}; \quad \Gamma' \to t_1 = t_2}{\Gamma^x_{t_2},\ \Gamma' \to \Delta^x_{t_2}}$$

R3. $$\frac{\Gamma \to \Delta}{\Gamma^x_t \to \Delta^x_t}$$

R4. $$e \to e$$

(3) Structural rules

R5. $$\frac{\Gamma \to \Delta}{\Gamma \to \Delta, e} \qquad \frac{\Gamma \to \Delta}{e, \Gamma \to \Delta} \text{ (weakening)}$$

R6. $$\frac{\Gamma \to \Delta, e; \quad e, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'} \text{ (cut) } \bigtriangledown$$

Remark 2.1.
(i) DEQ is sound and complete w.r.t. standard semantics (see [9] with reference to [1]).

(ii) DEQ represents a variant of the natural deduction system for the applied axiomatic theory of equations.
(iii) In DEQ the rule

$$\frac{\Gamma^x_{t_1} \to \Delta^x_{t_1}; \quad \Gamma' \to t_1 = t_2, \Delta'}{\Gamma^x_{t_2}, \Gamma' \to \Delta^x_{t_2}, \Delta'}$$

is derivable (using R2, R4 and R6).
(iv) From definition 2.2 it follows that DEQ implicitly contains the structural rules "exchange" and "contraction". $\triangledown$
In the following assume given a set of specific axioms (Sp. AX_k).
Definition 2.3 (the calculus DEQ'). A calculus DEQ' is constructed from DEQ by removing rule R3 and changing (Sp. AX_k) of DEQ in the following manner: let S be (Sp. AX_k) of DEQ, then $S^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}$ is the set of specific axioms (Sp. AX'_k) of DEQ', where $t_1, \ldots, t_n$ are arbitrary terms. $\triangledown$
Definition 2.4 (inclusion and equivalence of calculi). Let I_1, I_2 be arbitrary calculi then the notation $I_1 \Rightarrow I_2$ indicates that if $I_1 \vdash S$ then $I_2 \vdash S$ (where S is a sequent) and the notation $I_1 \equiv I_2$ means $I_1 \Rightarrow I_2$ and $I_2 \Rightarrow I_1$. $\triangledown$
Lemma 2.1. $DEQ \equiv DEQ'$.
Proof: Follows from the fact that $DEQ' \vdash \Gamma \to \Delta \Rightarrow DEQ' \vdash \Gamma^x_t \to \Delta^x_t$. $\triangledown$

3. Kanger-like calculus DEQ_1

Definition 3.1 (calculus DEQ_1). A calculus DEQ_1 is obtained from DEQ' by replacing axiom R4 and rule R2 with the rules (see e.g. [2]):

$$\frac{r = s, \Gamma^x_s \to \Delta^x_s}{r = s, \Gamma^x_r \to \Delta^x_r}\,(=_1) \qquad \frac{s = r, \Gamma^x_s \to \Delta^x_s}{s = r, \Gamma^x_r \to \Delta^x_r}\,(=_2)$$

Lemma 3.1. $DEQ_1 \equiv DEQ'$.
Proof: Follows from the following facts: (1) rules $(=_1), (=_2)$ are derivable in DEQ'; (2) R4 and R2 are derivable in DEQ_1. $\triangledown$

4. Rogava-like calculi DEQ_2, DEQ_3

Definition 4.1 (calculus DEQ_2). A calculus DEQ_2 is obtained from DEQ_1 by removing (cut) and replacing every specific axiom (Sp. AX'_k) by the following (Sp. cut$_k$) rule (see e.g. [8, 4]):

$$\frac{S_{11};\ldots;S_{1m};\ \ S_{21};\ldots;S_{2n}}{\Gamma\to\Delta}\ (\text{Sp. cut}_k),$$

where
$\Gamma\to\Delta := \Gamma_{11},\ldots,\Gamma_{1m},\Gamma_{21},\ldots,\Gamma_{2n}\to\Delta_{11},\ldots,\Delta_{1m},\Delta_{21},\ldots,\Delta_{2n}$

$$\begin{array}{l} S_{11} := \Gamma_{11}\to\Delta_{11},\ r_1=s_1 \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ S_{1m} := \Gamma_{1m}\to\Delta_{1m},\ r_m=s_m \\ S_{21} := u_1=v_1,\ \Gamma_{21}\to\Delta_{21} \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ S_{2n} := u_n=v_n,\ \Gamma_{2n}\to\Delta_{2n} \end{array}$$

$\bigtriangledown$

Definition 4.2 (calculus $I+\text{cut}$). Let I be some calculus not containing (cut), then $I+\text{cut}$ denotes the calculus obtained from I by adding (cut). $\bigtriangledown$

Lemma 4.1. (Sp. AX'_k) are derivable in DEQ_2.

Proof: Using rules (Sp. cut$_k$). $\bigtriangledown$

Lemma 4.2. The rules (Sp. cut$_k$) are derivable in DEQ_1.

Proof: Using the (cut) rule. $\bigtriangledown$

Theorem 4.1. $DEQ_1 \equiv DEQ_2 + (\text{cut})$.

Proof: From Lemmas 4.1 and 4.2. $\bigtriangledown$

Definition 4.3 (calculus DEQ_3). A calculus DEQ_3 is obtained from the calculus DEQ_2 by replacing rules $(=_1),(=_2)$ by the following rules (see e.g. [7]):

$$\frac{r=s,\ \Gamma\to\Delta^x_s}{r=s,\ \Gamma\to\Delta^x_r}\,(=_{1s}) \qquad \frac{s=r,\ \Gamma\to\Delta^x_s}{s=r,\ \Gamma\to\Delta^x_r}\,(=_{2s})$$

$\bigtriangledown$

Remark 4.1. Sometimes the notation $(=_s)$ will be used instead of the notation $(=_{is})$ $(i=1,2)$. $\bigtriangledown$

Lemma 4.3. $DEQ_3\vdash S \Rightarrow DEQ_2\vdash S$.

Proof. : Obvious. $\triangledown$
Definition 4.4. The notation $I \vdash^D S$ will represent the fact that one can construct a derivation D of the sequent S in calculus I. $\triangledown$
Lemma 4.4. $DEQ_2 \vdash^D S \Rightarrow DEQ_3 \vdash^{D^*} S.$
Proof. The proof is carried out by induction on $n\omega + h(D)$, where n is the number of antecedent applications of $(=_1), (=_2)$ and $h(D)$ is the height of the given derivation D. Consider the upper application of $(=_i)$ $(i = 1, 2)$ in D. We consider only the following case:

$$D^+ \left\{ \begin{array}{l} \dfrac{\dfrac{u = v,\ r = s(u),\ \Gamma^x_{s(u)} \to \Delta^x_{s(v)}}{u = v, r = s(u), \Gamma^x_{s(u)} \to \Delta^x_{s(u)}} (=_{2s})}{u = v,\ r = s(u),\ \Gamma^x_r \to \Delta^x_r} (=_1) \end{array} \right.$$

We construct the following derivation D' in DEQ_2 :

$$D' \left\{ \begin{array}{l} \dfrac{\dfrac{D_1 \left\{ \dfrac{u = v,\ r = s(u), \Gamma^x_{s(u)} \to \Delta^x_{s(v)}}{u = v,\ r = s(u), \Gamma^x_r \to \Delta^x_{s(v)}} (=_1) \right.}{u = v,\ r = s(u),\ \Gamma^x_r \to \Delta^x_{s(u)}} (=_{2s})}{u = v, r = s(u),\ \Gamma^x_r \to \Delta^x_r} (=_{1s}) \end{array} \right.$$

As $h(D_1) < h(D^+)$, then $DEQ_2 \vdash^{D_1} S_1 \Rightarrow DEQ_3 \vdash^{D_1^*} S_1$, where $S_1 := u = v, r = s(u), \Gamma^x_r \to \Delta^x_{s(v)}$ and $DEQ_2 \vdash^{D'} S_2 \Rightarrow DEQ_3 \vdash^{D'^*} S_2$, where $S_2 := u = v,\ r = s(u),\ \Gamma^x_r \to \Delta^x_r$. $\triangledown$
Theorem 4.2. $DEQ_2 \equiv DEQ_3$.
Proof: Follows from Lemmas 4.3 and 4.4. $\triangledown$
Remark 4.1. From the proof of Lemma 4.4 it follows that $h(D^*) > h(D)$, where D^* is the derivation of a sequent S in DEQ_3, D is the derivation of the same sequent S in DEQ_2. $\triangledown$
Lemma 4.5. $DEQ_3 \vdash r = r,\ \Gamma \to \Delta \Rightarrow DEQ_3 \vdash \Gamma \to \Delta$.
Proof: By induction on the height of the given derivation. $\triangledown$
Lemma 4.6. Consider the derivation D :

$$\frac{D_1 \{ \Gamma \to \Delta, e; \quad D_2 \{ e, \Pi \to \Theta}{\Gamma, \Pi \to \Delta, \Theta} \text{(cut)},$$

where D_1, D_2 are the derivations in DEQ_3, then $DEQ_3 \vdash \Gamma, \Pi \to \Delta, \Theta$.

Proof. The proof is carried out by induction on $h(D_1)$. Let $h(D_1) = 0$, then $\Gamma \to \Delta,\ e :=\to r = r$. Consider the derivation D_2 in DEQ_3. By Lemma 4.5 we get $DEQ_3 \vdash^{D_2'} \Pi \to \Theta$ and D_2' is the desired derivation. Let $h(D_1) > 0$, then we consider only the following case:

$$\dfrac{\dfrac{D_1'\{u = v,\ \Gamma \to \Delta,\ r = s(v)}{u = v,\ \Gamma \to \Delta,\ r = s(u)}(=_s) \quad D_2\{r = s(u),\ \Pi \to \Theta}{u = v,\ \Gamma, \Pi \to \Delta, \Theta}\text{(cut)}$$

Applying (weakening) and (=) to D_2 weget $DEQ_2 \vdash^{D_2'} S_2' := r = s(v),\ u = v,\ \Pi \to \Theta$. Using Lemma 4.4 we get $DEQ_3 \vdash^{D_2'^*} S_2'$. Applying (cut) to $D_1', D_2'^*$ we get $DEQ_3 + \text{cut} \vdash^{D'} S := u = v,\ \Gamma, \Pi \to \Delta, \Theta$. As $h(D_1') < h(D_1)$ we get $DEQ_3 \vdash S$. $\triangledown$

Theorem 4.3 (elemination theorem for DEQ_2). $DEQ_2 + \text{cut} \Rightarrow DEQ_2$.

Proof: Follows from Lemma 4.6 and Theorem 4.2. $\triangledown$

Theorem 4.4. $DEQ_1 \equiv DEQ_2$.

Proof: From Theorems 4.1 and 4.3. $\triangledown$

5. Specialization of (Sp. cut$_k$) rules

Definition 5.1 (calculus DEQ_4). A calculus DEQ_4 is obtained from DEQ_2 by replacing rules (Sp. cut$_k$) by the rules

$$\frac{S_{11}; \ldots; S_{1m}; S_{21}; \ldots; S_{2n}}{\Gamma \to \Delta}\ (\text{Sp.} =_k),$$

where

$\Gamma \to \quad \Delta :=$

$$\Gamma_{11}, \ldots, \Gamma_{1m}, \Gamma_{21}, \ldots, \Gamma_{2n} \to \Delta_{11}, \ldots, \Delta_{1m}, \Delta_{21}{}^{x_1}_{\tau_{11}}, \ldots, \Delta_{2n}{}^{x_n}_{\tau_{n1}}$$

$$S_{11} := \Gamma_{11} \to \Delta_{11},\ r_1 = s_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_{1m} := \Gamma_{1m} \to \Delta_{1m},\ r_m = s_m$$

$$S_{21} := \Gamma_{21} \to \Delta_{21}{}^{x_1}_{\tau_{12}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_{2n} := \Gamma_{2n} \to \Delta_{2n}{}^{x_n}_{\tau_{n2}}$$

where $\tau_{j1}, \tau_{j2} \in \{u_j, v_j\}$, $j \in \{1, \ldots, n\}$ and $\tau_{j1} \neq \tau_{j2}$. $\triangledown$

Remark 5.1. Special cases of (Sp. $=_k$) rules, corresponding to specific axioms of the form $\rightarrow u = v$, were presented in [8, 4, 5]. In [4, 5] the complex notion of variant of formula by specific axioms was used instead of the introduction of negative equalities $r_j = s_j$ $(j = 1, \ldots, m)$ of specific axioms as cut formulas in the premises $s_{11}, \ldots, s_{1m}$; equalities $u_j = v_j$ $(j = 1, \ldots, n)$ in the premises $s_{21}, \ldots, s_{2n}$ were introduced as cut formulas, besides. In rules (Sp. $=_k$) only the negative equalities of specific axioms are introduced as cut formulas. Therefore the complex notion of the variant of a formula is unnecessary. If $m = 0$ then the (Sp. $= k$) rules are purely cut-free. Analogous rules (corresponding to the case when $m = 0$) were presented in [6] for arithmetical robinson system and at the same time no cut-like rule was not introduced for the specific axiom containing a negative equality.

Example 5.1. Let DEQ' have the following three specific axioms (Sp. AX_k) $(k \in \{1, 2, 3\})$:

(1) $\quad r = s \rightarrow u_1(c_1) = v_1(c_2),\ u_2(c_3) = v_2(c_4)$

(2) $\quad u_1(c_5) = v_1(c_6) \rightarrow$

(3) $\quad u_2(c_7) = v_2(c_8) \rightarrow$

Then the calculus DEQ_4 has the following three (Sp. $=_k$) $(k \in \{1, 2, 3\})$ rules:

$$\frac{\Gamma_1 \rightarrow \Delta_1, r = s; \quad \Gamma_2 \rightarrow \Delta_2{}^{x_1}_{\tau_{12}}; \quad \Gamma_3 \rightarrow \Delta_3{}^{x_2}_{\tau_{22}}}{\Gamma_1, \Gamma_2, \Gamma_3 \rightarrow \Delta_1, \Delta_2{}^{x_1}_{\tau_{11}}, \Delta_3{}^{x_2}_{\tau_{21}}} \text{(Sp. } =_1)$$

$$\frac{\Gamma \rightarrow \Delta, u_1(c_5) = v_1(c_6)}{\Gamma \rightarrow \Delta} \text{(Sp. } =_2)$$

$$\frac{\Gamma \rightarrow \Delta, u_2(c_7) = v_2(c_8)}{\Gamma \rightarrow \Delta} \text{(Sp. } =_3),$$

where

$$\tau_{11}, \tau_{12} \in \{u_1(c_1), v_1(c_2)\}$$
$$\tau_{21}, \tau_{22} \in \{u_2(c_3), v_2(c_4)\}$$

Let $S := \Gamma \rightarrow$, where $\Gamma := s = r$, $c_1 = c_5, c_2 = c_6, c_3 = c_7, c_4 = c_8$, then applying (Sp. $=_3$), (Sp. $=_2$), ($=_s$), (Sp. $=_1$) bottom-up we get $DEQ_4 \vdash S$. $\triangledown$

Lemma 5.1. $DEQ_4 \Rightarrow DEQ_3$.
Proof: Follows from the fact that rules (Sp. $=_k$) are derivable in DEQ_3. $\bigtriangledown$
Example 5.2. Let DEQ_4 be as in Example 5.1. Consider the rule (Sp. $=_1$). Let us show that (Sp. $=_1$) is derivable in DEQ_3. Let
$S_1 := \Gamma_1 \to \Delta_1, r = s$
$S_2 := \Gamma_2 \to \Delta_2\,{}^{x_1}_{\tau_{12}}$
$S_3 := \Gamma_3 \to \Delta_3\,{}^{x_2}_{\tau_{22}}$
Applying (W), $(=_{is})$ $(i = 1, 2)$ to S_2, S_3 we get
$S_2' := u_1(c_1) = v_1(c_2), \Gamma_2 \to \Delta_2\,{}^{x_1}_{\tau_{11}}$
$S_3' := u_2(c_3) = v_2(c_4), \Gamma_3 \to \Delta_3\,{}^{x_2}_{\tau_{21}}$
Applying (Sp. cut$_1$) to S_1, S_2', S_3' we get the conclusion S of (Sp.$=_1$). $\bigtriangledown$
Lemma 5.2. Consider the derivation D

$$\dfrac{\overset{\vdots}{S_{11}};\ldots;\overset{\vdots}{S_{1m}};\overset{\vdots}{S_{21}};\ldots;\overset{\vdots}{S_{2n}}}{\Gamma \to \Delta}\ (\text{Sp. cut}_k)$$

and let $D_{11}, \ldots, D_{1m}, D_{21}, \ldots, D_{2n}$ be the derivations of the sequents $S_{11}, \ldots, S_{1m}, S_{21}, \ldots, S_{2n}$, respectively, in the calculus DEQ_4, then $DEQ_4 \vdash \Gamma \to \Delta$.
Proof: By induction on $h(D_{21}) + \cdots + h(D_{2n})$. $\bigtriangledown$
Example 5.3. Let DEQ_4 be the same as in Example 5.1 and let S_1, S_2', S_3' be the same as in Example 5.2. Then (Sp. cut$_1$) has the following form:

$$\dfrac{D_1\{S_1;\ D_2\{S_2';\ D_3\{S_3'}{S}\ (\text{Sp.cut}_1)$$

Let us consider only the case when D_2, D_3 have the following forms:

$$D_2\left\{\begin{array}{l} D_2'\Big\{\dfrac{u_1(c_1) = v_1(c_2),\ \Gamma_2 \to \Delta_2\,{}^{x_1}_{\tau_{12}}}{u_1(c_1) = v_1(c_2),\ \Gamma_2 \to \Delta_2\,{}^{x_1}_{\tau_{11}}}\,(=_{is}) \end{array}\right.$$

$$D_3\left\{\begin{array}{l} D_3'\Big\{\dfrac{u_2(c_3) = v_2(c_4),\ \Gamma_3 \to \Delta_3\,{}^{x_2}_{\tau_{22}}}{u_2(c_3) = v_2(c_4),\ \Gamma_3 \to \Delta_3\,{}^{x_2}_{\tau_{21}}}\,(=_{is}) \end{array}\right.$$

Having set $\Gamma := \Gamma_1,\ \Gamma_2,\ \Gamma_3$ and $S_2 := u_1(c_1) = v_1(c_2),\ \Gamma_2 \to \Delta_2{}^{x_1}_{\tau_{12}}$, $S_3 := u_2(c_3) = v_2(c_4),\ \Gamma_3 \to \Delta_3{}^{x_2}_{\tau_{22}}$, we construct the following derivation D^*:

$$D^* \left\{ \dfrac{S_1; \quad \dfrac{S_1;\ D'_2\{S_2;\ D_3\{S'_3}{\Gamma \to \Delta_1, \Delta_2{}^{x_1}_{\tau_{12}}, \Delta_3{}^{x_2}_{\tau_{21}}} \quad \dfrac{S_1;\ D_2\{S'_2;\ D'_3\{S_3}{\Gamma \to \Delta_1, \Delta_2{}^{x_1}_{\tau_{11}}, \Delta_3{}^{x_2}_{\tau_{22}}}}{\Gamma \to \Delta_1, \Delta_2{}^{x_1}_{\tau_{11}}, \Delta_3{}^{x_2}_{\tau_{21}}} \text{(Sp. } =_1) \right.$$

As $h(D'_2) < h(D_2)$ and $h(D'_3) < h(D_3)$, The (Sp. cut_1) steps (unlabeled above) can be eliminated from the derivation D^*.

Theorem 5.1. $DEQ_4 \equiv DEQ_3$.

Proof: Follows from Lemmas 5.1 and 5.2. $\bigtriangledown$

Definition 5.2 (calculus DEQ_5). A calculus DEQ_5 is obtained from calculus DEQ_4 by replacing rules (Sp. $=_k$) by the rules (Sp. $=^*_k$) :

$$\frac{S_{11}; \ldots ; S_{1m};\ S_{21}; \ldots ; S_{2n}}{\Gamma \to \Delta} \text{(Sp. } =^*_k),$$

where

$$\begin{aligned}
&\Gamma \to \Delta := \Gamma \to \Delta{}^{x_1 \ldots x_n}_{\tau_{11} \ldots \tau_{n1}} \\
&S_{11} := \Gamma \to \Delta{}^{x_1 \ldots x_n}_{\tau_{11} \ldots \tau_{n1}}, r_1 = s_1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&S_{1m} := \Gamma \to \Delta{}^{x_1 \ldots x_n}_{\tau_{11} \ldots \tau_{n1}}, r_m = s_m \\
&S_{21} := \Gamma \to \Delta{}^{x_1 x_2 \ldots x_n}_{\tau_{12} \tau_{21} \ldots \tau_{n1}} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&S_{2n} := \Gamma \to \Delta{}^{x_1 \ldots x_{n-1} x_n}_{\tau_{11} \ldots \tau_{n-11} \tau_{n2}}
\end{aligned}$$

$\bigtriangledown$

Lemma 5.2. $DEQ_4 \equiv DEQ_5$.

Proof: Using structural rule weakening. $\bigtriangledown$

Definition 5.3 (calculus DEQ^*). A calculus DEQ^* is obtained from the calculus DEQ_5 by dropping structural rule weakening and replacing the axiom $\to r = r$ by the axiom $\Gamma \to \Delta, r = r$. $\bigtriangledown$

Lemma 5.3. Structural rule weakening is admissible in DEQ^*.
Proof: By induction on the height of a given derivation. $\triangledown$
Theorem 5.2. $DEQ^* \equiv DEQ$.
Proof: Follows from Lemmas 2.1, 3.1, Theorems 4.2, 4.4, 5.1 and Lemmas 5.2, 5.3. $\triangledown$

6. Some specializations of (Sp. $=_k$) rules

Definition 6.1 (contrary calculus DEQ). A calculus DEQ will be called *contrary* if there exists some substitution $\bar{x}/\bar{t}$ (where $\bar{x} := x_1, \ldots, x_n$; $\bar{t} := t_1, \ldots, t_n$) and some specific axiom k containing a negative occurrence of an equality $r_i^k = s_i^k$, where $i \in \{1, \ldots, m_k\}$, such that using only equality rules $(=_{is})$ it is possible to construct a derivation of the sequent $\Gamma_{\bar{t}}^{\bar{x}} \to (r_i^k = s_i^k)_{\bar{t}}^{\bar{x}}$ where Γ consists of positive occurrences of equality in specific axioms of DEQ and maybe some equalities constituted only from symbols of constants entering in axioms (Sp. AX_k) In the opposite case DEQ will be called *non-contrary.* $\triangledown$
Example 6.1. Let DEQ have the following specific axioms (Sp. AX_k) $(k \in \{1, 2, 3\})$:
(1) $\quad f(u) = r(a) \quad \to t_1 = t_2(c_1)$
(2) $\quad \to f(c_1) = g(v)$
(3) $\quad \to r(b) = g(c_2)$
where a, b, c_1, c_2 are some constants.
DEQ is contrary because it is possible to construct the derivation D of the sequent $\Gamma_{\bar{t}}^{\bar{x}} \to (r_i^k = s_i^k)_{\bar{t}}^{\bar{x}}$, where $\Gamma := f(c_1) = g(v),\ r(b) = g(c_2),\ a = b$; $(r_i^k = s_i^k) := f(u) = r(a)$ and the substitution $\bar{x}/\bar{t}$, where $\bar{x} := u, v$; $\bar{t} := c_1, c_2$. The derivation D has the form:

$$\cfrac{\cfrac{\cfrac{f(c_1) = g(c_2),\ r(b) = g(c_2),\ a = b \to r(a) = r(a)}{f(c_1) = g(c_2),\ r(b) = g(c_2),\ a = b \to r(b) = r(a)}\,(=_s)}{f(c_1) = g(c_2),\ r(b) = g(c_2),\ a = b \to g(c_2) = r(a)}\,(=_s)}{f(c_1) = g(c_2),\ r(b) = g(c_2),\ a = b \to f(c_1) = r(a)}\,(=_s)$$

$\triangledown$
Example 6.2. Let DEQ has the following (Sp. AX_k) $(k \in \{1, 2\})$:
(1) $\quad f(u) = g(v) \quad \to t_1 = t_2(c_1)$
(2) $\quad \to g(c_1) = f(c_2)$

where c_1, c_2 are some constants.
It is easy to see that DEQ is contrary under the substitution $\bar{x}/\bar{t}$, where $\bar{x} := u, v; \bar{t} := c_2, c_1$. The following derivation of the sequent $y = c_1,\ u = c_2,\ v = c_1 \to t_1 = t_2(y)$ shows the necessity of introducing as cut formula the negative occurrence of equality $f(u) = g(v)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \to f(c_2) = f(c_2)}{\Gamma \to f(c_2) = g(c_1)}\,(\text{Sp.} =_2)}{\Gamma \to f(u) = g(v)}\,(=_s),(=_s) \qquad \Gamma \to t_1 = t_1}{\Gamma \to t_1 = t_2(c_1)}\,(\text{Sp.} =_1)}{\Gamma \to t_1 = t_2(y)}\,(=_s)$$

where $\Gamma := y = c_1,\ u = c_2,\ v = c_1$. $\bigtriangledown$
Definition 6.2 (calculus DEQ^{**}). The calculus DEQ^{**} is obtained from the calculus DEQ^* (1) replacing rules (Sp. $=^*_k$) by following ones:

$$\frac{S'_{21}; \ldots ; S'_{2n}}{\Pi, \Gamma \to \Delta^{x_1 \ldots x_n}_{\tau_{11} \ldots \tau_{n1}}}\,(\text{Sp.} =^{**}_k),$$

where $\Pi := r'_1 = s'_1, \ldots , r'_m = s'_m;\quad r'_i \in \{r_i, s_i\},\ s'_i \in \{r_i, s_i\},\ i \in \{1, \ldots , m\},\ r'_i \neq s'_i$; S'_{2j} is obtained from S_{2j} of rules (Sp. $=^*_k$) changing Γ by Π, Γ; (2) replacing rules $(=_{is})$ by $(=_i)$ $(i = 1, 2)$. $\bigtriangledown$
Theorem 6.1. Let DEQ be a non-contrary calculus, then $DEQ \equiv DEQ^{**}$.
Proof: Applying the definition of contrary calculus. $\bigtriangledown$
Example 6.3. Let DEQ be the following (Sp. AX_k) $(k \in \{1, 2\})$:
(1) $\quad g(v) = f(u) \quad \to t_1 = t_2(c_1)$
(2) $\quad \to g(c_1) = g(c_2)$
where c_1, c_2 are some constants.
It is easy to see that DEQ is non-contrary. We construct the following derivations of the sequent $y = c_1,\ v = s, u = r,\ f(u) = g(s) \to t_1 = t_2(y)$, which shows how the rule (Sp. $=^{**}$)works :

$$\cfrac{\cfrac{\cfrac{\Gamma \to t_1 = t_1}{\Gamma \to t_1 = t_2(c_1)}\,(\text{Sp.} =^{**})}{\Gamma \to t_1 = t_2(y)}\,(=_s)}{\Gamma^* \to t_1 = t_2(y)}\,(=)$$

where $\Gamma := y = c_1,\ v = s, u = r, f(u) = g(v);\ \Gamma^* := y = c_1, v = s, u = r, f(u) = g(s)$. $\bigtriangledown$

Remark 6.1. To prove that $DEQ \equiv DEQ^{**}$ in case DEQ is contrary it is necessary to change the rules (Sp. $=_k^{**}$) in such a way that they involve the notion of variant of formula by specific axioms (see [4,5]). $\triangledown$

7. Elimination of contraction and exchange

Definition 7.1 (calculus DEQ_6). A calculus DEQ_6 is obtained from the calculus DEQ^* in the following way:
(i) the Γ, Δ in sequent $\Gamma \to \Delta$ are considered arbitrary lists rather than sets;
(ii) adding the structural rules of contraction and exchange:

$$\frac{\Gamma, e_1, e_2, \Pi \to \Delta}{\Gamma, e_2, e_1, \Pi \to \Delta}\,(EX \to) \qquad \frac{\Gamma \to \Delta, e_1, e_2, \Theta}{\Gamma \to \Delta, e_2, e_1, \Theta}\,(\to EX)$$

$$\frac{e, e, \Gamma \to \Delta}{e, \Gamma \to \Delta}\,(C \to) \qquad \frac{\Gamma \to \Delta, e, e}{\Gamma \to \Delta, e}\,(\to C)$$

$\triangledown$
Lemma 7.1. $DEQ_6 \equiv DEQ^*$.
Proof: Applying definition 7.1. $\triangledown$
Definition 7.2. (calculus DEQ_7). The calculus DEQ_7 is obtained from the calculus DEQ_6 by dropping structural rules $(EX \to), (\to EX)$. $\triangledown$
Lemma 7.2. $DEQ_6 \equiv DEQ_7$.
*Proof:*d Aanalogously to the proof in [7]. $\triangledown$
Definition 7.3 (Calculus DEQ_8). A calculus DEQ_8 is obtained from the calculus deq_7 by dropping the structural rules $(C \to), (\to C)$ and changingthe rules $(=)$ and (Sp. $=_k$) in such a way that the main formulas are duplicated in the premises of the rule. $\triangledown$
For example, instead of the rule $(=_{1s})$ the following rule:

$$\frac{r = s, \Gamma \to \Delta_s^x, \Delta_r^x}{r = s, \Gamma \to \Delta_r^x}\,(=_{1s}^*)$$

is in the calculus DEQ_8.
Definition 7.4 (calculus DEQ_7'). A calculus DEQ_7' is obtained from the calculus DEQ_7 by dropping the structural rule $(C \to)$. $\triangledown$
Lemma 7.3. $DEQ_7 \vdash^D S \Rightarrow DEQ_7' \vdash^{D^*} S$.

Proof: By induction on $n\omega + h(D)$, where n is the number of applications of $(C \to)$ in D. $\bigtriangledown$

Lemma 7.4. $DEQ'_7 \vdash^{D} \Rightarrow DEQ_8 \vdash^{D^*} S$.

Proof. The proof is carried out by $n\omega + h(D)$, where n is the number of applications of $(\to C)$ in D. We consider only the following case

$$D' \left\{ \begin{array}{l} D'_1 \left\{ \dfrac{\dfrac{\Gamma,\ r = s \to \Delta,\ p(s) = q, p(r) = q, p(r) = q}{\Gamma,\ r = s \to \Delta,\ p(r) = q,\ p(r) = q}\,(=^*_s)}{\Gamma,\ r = s \to \Delta,\ p(r) = q}\,(\to C) \right. \end{array} \right.$$

where D'_1 is the derivation in DEQ_8. Let us construct the following derivation D'^* :

$$D'^* \left\{ \begin{array}{l} D'^*_1 \left\{ \dfrac{\dfrac{\Gamma,\ r = s \to \Delta,\ p(s) = q, p(r) = q, p(r) = q}{\Gamma,\ r = s \to \Delta,\ p(s) = q,\ p(r) = q}\,(\to C)}{\Gamma,\ r = s \to \Delta,\ p(r) = q}\,(=^*_s) \right. \end{array} \right.$$

As $h(D'^*_1) < h(D'_1)$ then $DEQ'_7 \vdash^{D'^*} S^* \Rightarrow DEQ_8 \vdash^{D'^{**}} S^*$, where $S^* := \Gamma, r = s \to \Delta, p(r) = q$. $\bigtriangledown$

Theorem 7.2. $DEQ_7 \equiv DEQ_8$.

Proof: Follows from Lemmas 7.3 and 7.4. $\bigtriangledown$

Remark 7.1. Let $p = q$ be any explicitly indicated term in (Sp. AX_k) of DEQ', then p, q will be called *axiom terms*. We introduce an ordering of terms: Let $r(t)$ stand for the rank of a term t which is defined in the following way:

(1) $r(t) = 0$, if t is any axiom term;

(2) $r(t_1) < r(t_2)$, if t_1, t_2 are not axiom terms and t_1 is less then t_2 in a lexicograpfic ordering of terms.

Then using the ordering of terms introduced above we can drop duplication of the main formulas in the equality rules and leave this duplication in the rules (Sp. $=^*_k$) only. $\bigtriangledown$

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