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THE DEFINABILITY OF TRUTH IN A LOGIC OF SENTENTIAL OPERATORS

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Abstract. Logic of Sentential Operators, LSO, extends classical logic with sentential quantifiers and operators, making self-reference and paradoxes expressible. All classical tautologies and contradictions remain intact, while the reasoning system preserves the full rule set of the classical sequent calculus. The paper defines operators that capture in LSO its semantic structure and truth. The extension with these operators is conservative, while the truth operator satisfies the unrestricted Convention T and the compositionality axioms. Its definition uses the operator of syntactic equality of sentences. A proof of its consistency is provided, thereby filling a gap in earlier presentations of LSO.

§1. Introduction. Truth theories study truth predicates in a sufficiently strong arithmetic with arithmetised syntax, so that terms both denote their semantic referents and code formulas by Gödelisation. The technical solutions can vary, so by "AST" we refer generally to such frameworks that represent the metalanguage by terms of the object-language and truth by a predicate. Their well-known limitations, arising from Tarski's undefinability theorem, pose the annoying choice: either (a) working in AST, investigate various restrictions on Convention T that avoid paradoxes, or (b) seek an alternative framework capable of defining its truth.

Option (a) has yielded impressive results but, philosophically, seems to continue exploring its own limitations rather than the broader landscape of reasoning. The claim that natural language "for which the normal laws of logic hold, must be inconsistent" [5, p.165], or else cannot fully articulate its own truth concept, is perhaps imaginable but not compelling. "[S]omehow, it seems, natural languages defy the indefinability theorem", [4], and constraining the AST notion of truth to formalised languages is just a prudent philosophical retreat.

The theorem holds for classical logic under some minimal assumptions about the language, so increasing expressivity offers no help. Option (b) thus effectively abandons classical logic – an unattractive step, usually accompanied by attempts to retain as much of it as possible. This paper does not go that far. Instead of changing the logic, it changes the model of self-reference, to one not using Gödelisation. A Cretan's remark "All Cretans always lie" involves a kind of material supposition, a reference to the sentence itself rather than only to its truth-value, without naming it. Addressing sentences through their names – technically convenient and elegant – is not the only mode of metalinguistic reference. This is especially significant considering that Tarski proved the undefinability of the truth predicate, not of truth.

In the logic of sentential operators, LSO [8], statements apply to statements by means of such operators, rather than predicates on sentence names.[1] A sentence in a *sentential position* (i.e., not in the scope of any operator) stands for its truth-value, whereas one occurring in a *nominal position*, as the argument of an operator, may be taken in material supposition. We can quantify over both occurrence types, e.g., in $\forall\phi(K(\phi) \to \neg\phi)$ the first occurrence of ϕ is nominal and the

[1] The formal difference between predicate and operator need not reflect any specific linguistic difference. The truth operator may be read as saying "it is true that...", but the possible correspondence to the expressions of informal language need not be tight. To the extent the considered formalisms reflect natural language, they do so in its logical and semantic aspects rather than the grammatical or syntactic ones.

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second one sentential. Reading $K(\phi)$ as 'Cretan K saying ϕ', this stands for 'everything Cretan K says is false', and $K(\forall\phi(K(\phi) \to \neg\phi))$ for Cretan K saying that.

The metalanguage, marked by the operators, is thus part of the language. Paradoxes in LSO are metalinguistic claims arising from the unfortunate valuations of operators that fail to extend to the entire language. Their expressibility requires no artificial restrictions to maintain consistency – just the usual care in formulating one's claims. The liar paradox, arising when K says only $\forall\phi(K(\phi) \to \neg\phi)$, is resolved if K says also something true. Like the (in)consistency of a theory is the responsibility of its designer, paradoxes arise from unsatisfiable claims and are not excluded by the language or the formalism alone. Building on this earlier treatment of paradoxes, the paper formalises truth of LSO sentences in LSO. The transition from predicates to operators circumvents the earlier dilemma between (a) classical logic and (b) a possible definability of truth. LSO is classical in the sense of being two-valued and retaining all rules of the classical sequent system, augmenting it with two rules for the sentential quantifiers. The semantics is formulated differently to cater for the self-reference present in the language but, when restricted to the sublanguage without the sentential quantifiers and operators, is equivalent to its classical semantics.

Convention T – formulated as (T) $\forall\phi(\mathcal{T}\phi \leftrightarrow \phi)$ – acts in the way proposed by Tarski: as the necessary condition of material adequacy for a definition of truth. In the operator setting, it can be taken as a definition introducing a trivial, identity-like operator. A satisfactory definition, however, should possess some substance, even if it is unclear what exactly this means.[2] One would naturally expect it to reflect how the semantic structure makes some sentences true and others false. The paper provides such a definition, but without offering any sufficient criteria. Any consistent theory that implies (T) might qualify as a candidate whose merits must be assessed on broader grounds. Hopefully, the reader will agree that the proposed definition improves significantly over bare (T). It remains to be seen if others can outperform it.

Any classical language $\mathcal{L}^-$ (FOL is used as the natural example) is expanded to $\mathcal{L}^{\doteq}$ with the sentential quantifiers and operator $\doteq$ holding for syntactically identical sentences, and then to $\mathcal{L}$ with any operators. A digraph $G_M(\mathcal{L})$ provides the interpretation of $\mathcal{L}$ which, restricted to $\mathcal{L}^-$, coincides with its classical interpretation in a FOL domain M. Adding operators for the edge relation and truth yields language $\mathcal{L}^{\mathcal{T}}$ in which truth is defined by schemas for all $\mathcal{L}^{\mathcal{T}}$ formulas. Its uniform formulation – for $G_M(\mathcal{L}^{\mathcal{T}})$ over all domains M – captures validity of $\mathcal{L}^{\mathcal{T}}$ sentences. Truth in a given structure is definable if the structure is axiomatisable in the object-language $\mathcal{L}^-$.

Some elements differ from the earlier presentations of LSO, [7, 8], so Section 2 provides the required material making the paper technically self-contained. The definition of truth relies only on the operator $\doteq$ of syntactic identity of sentences, whose consistency was earlier assumed but is now proven for the first time. Section 3 presents the main contribution and Section 4 some concluding philosophical remarks.

§2. The background: LSO. In addition to the elements of a classical (here, FOL) language $\mathcal{L}^-$ – object variables $o\mathcal{V}$ (typically, x, y), function and constant symbols $\boldsymbol{Fn}$, predicate symbols $\boldsymbol{Pr}$ – an LSO language $\mathcal{L}$ has also sentential operator symbols $\boldsymbol{Op}$ and sentential variables $s\mathcal{V}$ (typically, ϕ, ψ), which can be quantified. The language is given by the following grammar in BNF. (Single arguments stand for arbitrary arities; $(\boldsymbol{T}/\boldsymbol{F})$ abbreviate all combinations of term/formula arguments, including empty ones for the constants; $\mathcal{V} = o\mathcal{V} \cup s\mathcal{V}$.)

$$\begin{array}{lll} \boldsymbol{T} & ::= & o\mathcal{V} \mid \boldsymbol{Fn}(\boldsymbol{T}) & \text{– terms} \\ \boldsymbol{A} & ::= & \boldsymbol{Pr}(\boldsymbol{T}) \mid \boldsymbol{Op}(\boldsymbol{T},\boldsymbol{F}) \mid s\mathcal{V} & \text{– atomic formulas} \\ \boldsymbol{F} & ::= & \boldsymbol{A} \mid \boldsymbol{F} \wedge \boldsymbol{F} \mid \neg\boldsymbol{F} \mid \forall\mathcal{V}.\boldsymbol{F} & \text{– all formulas.} \end{array}$$

Other connectives are used with the classical definitions. Formulas with possibly free $X \subset \mathcal{V}$ are denoted $\boldsymbol{F}_X$. Sentences, i.e., formulas without free variables $\boldsymbol{F}_\varnothing$, are denoted by $\boldsymbol{S}$. Among the operators $\boldsymbol{Op}$, we include the binary infix operator $\doteq$ of syntactic identity of sentences, *s-equality*. A *trivial* equality is $F \doteq F$, for any formula F. Generally, operators bind stronger than logical connectives, but we use parentheses to disambiguate, e.g., $(\phi \doteq A) \to B$ versus $\phi \doteq (A \to B)$.

[2] Deflationists might disagree. Our notion is conservative, but we refrain from discussing whether this complies with deflationism or goes beyond it.

The operators are sentential but, syntactically, can have open formulas as arguments. If $F(x,\phi)$ has only x and ϕ free, then $O(F(x,\phi))$ is an open formula with these variables free, while $\exists\phi\forall x O(F(x,\phi))$ is a sentence with a legal application of operator O.

An application of an operator to any argument(s) is an atomic expression. Hence, arbitrary valuations of such closed atoms are admissible, e.g., $O(S)$ and $O(\neg S)$, for a sentence S, may obtain all four possible combinations of truth values. Specific interactions of the operators with their arguments are left to the appropriate axiomatisations, which are not addressed here.

We view an operator application $O(S)$ as statement O about sentence S. Likewise, $O(O(S))$ states O about $O(S)$. Applications of operators, together with the sentential quantifiers, form thus the metalanguage – not only for the underlying object-language $\mathcal{L}^-$, but for $\mathcal{L}$ itself.

Sentential quantifiers suggest that sentences are among the objects of the interpretation domain, but this involves an additional structure. By $\mathcal{L}_M$ we denote the language $\mathcal{L}$ expanded with object-level constant symbols M, while by $\boldsymbol{T}_M, \boldsymbol{A}_M, \boldsymbol{S}_M$ – the free algebras of the respective syntactic categories over the elements of set M. Sentential quantifiers are interpreted substitutionally with the unrestricted substitution class of all $\boldsymbol{S}_M$, including all $\boldsymbol{S}$. Thus, in any sentence such a quantifier ranges also over this very sentence.

Typically, M is a *domain*, namely, a nonempty set with an interpretation of $\mathcal{L}^-$ function symbols, but not of the predicate or operator symbols. (In $\boldsymbol{T}_M/\boldsymbol{A}_M/\boldsymbol{S}_M$ we identify then each closed term of $\mathcal{L}^-$ with $m \in M$ it denotes. The special case $M = \varnothing$ for $\mathcal{L}^- = \varnothing$, of quantified boolean formulas, is admitted.) The ways in which operators acting on terms can affect the domain may require further investigation. Here, they are applied only to sentences $\boldsymbol{S}_M$, so M can be restricted at most by the (axioms of the) object-language $\mathcal{L}^-$. The phrase "for any domain" refers only to the domains that satisfy any such actual restrictions (which are not addressed in the paper).

A *valuation of variables* in M is a mapping $\alpha \in M^{o\mathcal{V}} \times (\boldsymbol{S}_M)^{s\mathcal{V}}$ of o-variables to the elements of M and s-variables to sentences $\boldsymbol{S}_M$. Each domain M determines a digraph over vertex set $\boldsymbol{S}_M$.

Definition 2.1. *The* language graph $G_M(\mathcal{L})$ *has* $\boldsymbol{S}_M$ *as vertices and edges given by:*

1. *for each atomic sentence* $A \in \boldsymbol{A}_M$*, both literals form a 2-cycle:* $A \leftrightarrows \neg A$*;*

while each non-atomic sentence $S \in \boldsymbol{S}_M$ *is the root of the subgraph* $G_M(S)$*:*

	root S	*with edges*	*to* :
2.	$\neg F$	$\longrightarrow$	F,
3.	$F_1 \wedge F_2$	$\longrightarrow$	$\neg F_i$, *for* $i \in \{1,2\}$,
4*a*.	$\forall x F x$	$\longrightarrow$	$\neg F(m)$, *for each* $m \in M$,
4*b*.	$\forall\phi F\phi$	$\longrightarrow$	$\neg F(T)$, *for each* $T \in \boldsymbol{S}_M$.

In 4b, all sentences $\boldsymbol{S}_M$ instantiate the s-quantifier, not only sentences $\boldsymbol{S}$ of $\mathcal{L}$. Consequently, the graph does not depend on the closed $\mathcal{L}^-$ terms, e.g., $G_M(\mathcal{L}) = G_M(\mathcal{L}_M)$. We drop M when it is inessential. By $\mathcal{LG}r(\mathcal{L})$ we denote all language graphs for a language $\mathcal{L}$. For each $\mathcal{L}^-$ sentence S its subgraph $G(S)$, rooted in S, is a tree except that instead of the leaves it has 2-cycles with the literals. More complex cycles arise from the s-quantification as illustrated further down.

In a language graph $G = (\boldsymbol{V}, \boldsymbol{E})$, vertices are assigned truth values following the rule

$$\forall x \in \boldsymbol{V} : \alpha(x) = \mathbf{1} \Leftrightarrow \forall y(\boldsymbol{E}(x,y) \to \alpha(y) = \mathbf{0}). \tag{2.2}$$

For $Y \subseteq \boldsymbol{V}$, let $\boldsymbol{E}^-(Y) = \{x \in \boldsymbol{V} \mid \exists y \in Y : \boldsymbol{E}(x,y)\}$ denote the set of vertices with an edge to some $y \in Y$. Vertices assigned $\mathbf{1}$ in (2.2) form a *kernel* of G, namely, a subset $K \subseteq \boldsymbol{V}$, that is *independent*, $\boldsymbol{E}^-(K) \subseteq \boldsymbol{V} \setminus K$ (no edges between vertices in K; equivalently, $\boldsymbol{E}(K) \subseteq \boldsymbol{V} \setminus K$), and *absorbing*, $\boldsymbol{E}^-(K) \supseteq \boldsymbol{V} \setminus K$ (each vertex outside of K has an edge to K). Conversely, for any kernel K, valuation $\kappa(x) = \mathbf{1} \Leftrightarrow x \in K$ satisfies (2.2). So we identify the two and denote by $Ker(G)$ all kernels (valuations (2.2)) of a graph G.

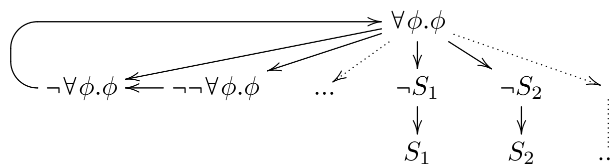
A kernel K of a language graph G, viewed as the set of true sentences, restricted to atoms determines their valuation κ. The object-level sentences in K are then exactly those that are true in $\mathcal{L}^-$ structure (M,κ). The semantics of $\mathcal{L}$ defined by kernels of language graphs, when restricted to the object-language $\mathcal{L}^-$, coincides with its classical semantics.

An LSO $\mathcal{L}$ *structure* is a pair (G,K) with $G \in \mathcal{LG}r(\mathcal{L})$ and $K \in Ker(G)$. (As K determines G, we often drop the latter.) A *kernel model* of a theory $\Gamma \subseteq \boldsymbol{S}$ is an $\mathcal{L}$ structure with a kernel containing all sentences from Γ. This is the first equivalence below, the second one reflecting the definition of a kernel and (2.2).

$$(G,K) \vDash_c S \Leftrightarrow S \in K \Leftrightarrow \forall\phi(\boldsymbol{E}(S,\phi) \to \phi \notin K). \tag{2.3}$$

Every FOL $\mathcal{L}^-$ structure M gives rise to some LSO $\mathcal{L}$ structure (G_M, K) satisfying the same $\mathcal{L}^-$ sentences. The extension of $\mathcal{L}^-$ to $\mathcal{L}$ is conservative: all classical tautologies (contradictions) remain tautologies (contradictions), sometimes expressible by single formulas, e.g., $\forall\phi(\phi \vee \neg\phi)$ is a tautology. The richer language introduces also some new tautologies and contradictions.

Example 2.4. *Let $S_1, S_2, \ldots$ stand for all sentences $\boldsymbol{S}_M$, except the iterated negations of $\forall\phi.\phi$, (some shown on the drawing of the graph $G(\forall\phi.\phi)$):*



The simplified drawing indicates only the relevant elements. Any $S_i \in \boldsymbol{S}_M$ valuated to $\mathbf{0}$ yields $\neg S_i = \mathbf{1}$ and $\forall\phi.\phi = \mathbf{0}$, but even if all $S_i = \mathbf{1}$, merely the indicated cycles force $\forall\phi.\phi = \mathbf{0}$. To obtain a kernel, the odd cycle via $\neg\neg\forall\phi.\phi$ must be broken, i.e., some of its vertices must have an out-neighbour $= \mathbf{1}$. If all $\neg S_i = \mathbf{0}$, this still happens when $\neg\forall\phi.\phi = \mathbf{1}$, making $\neg\neg\forall\phi.\phi = \mathbf{0} = \forall\phi.\phi$. In a sense, contradiction $\forall\phi.\phi$ is a counterexample to its own satisfiability.

Kernel semantics has a natural generalisation. Every kernel is also a *semikernel* [3], namely, a subset L of vertices such that

$$\boldsymbol{E}(L) \subseteq \boldsymbol{E}^-(L) \subseteq \boldsymbol{V} \setminus L. \tag{2.5}$$

Semikernels of G are denoted by $SK(G)$. Semikernel retains condition (2.2) for vertices it *covers*, namely, $\boldsymbol{E}^-[L] = L \cup \boldsymbol{E}^-(L)$, and satisfies vacuously formulas it does not cover, i.e., for $S \in \boldsymbol{S}_M$:

$$L \vDash S \Leftrightarrow (S \notin \boldsymbol{E}^-[L] \vee S \in L). \tag{2.6}$$

A *semikernel model* of a theory Γ is a semikernel (of a language graph) containing Γ. The following adaptation of this notion, for use with sequents (i.e., pairs of sets of formulas, denoted $\Gamma \Rightarrow \Delta$), admits extensions $\mathcal{L}^+$ of $\mathcal{L}$ with any sets of s-constants (needed in the proof of completeness).

Definition 2.7. *For an $\mathcal{L}$ sequent $\Gamma \Rightarrow \Delta$: a semikernel L of a graph $G_M \in \mathcal{LG}r(\mathcal{L}^+)$* models *it, $L \vDash \Gamma \Rightarrow \Delta$, iff L satisfies it under every valuation $\alpha \in M^{o\mathcal{V}} \times (\boldsymbol{S}^+_M)^{s\mathcal{V}}$ of free variables of Γ, Δ:*

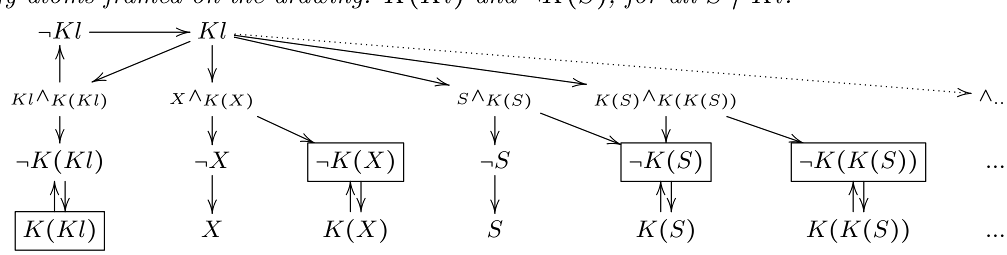
$$\alpha(\Gamma) \cup \alpha(\Delta) \subseteq \boldsymbol{E}^-[L] \Rightarrow \alpha(\Gamma) \cap \boldsymbol{E}^-(L) \neq \varnothing \vee \alpha(\Delta) \cap L \neq \varnothing. \tag{*}$$

$\Gamma \Rightarrow \Delta$ is valid, *$\Gamma \vDash \Delta$, iff $L \vDash \Gamma \Rightarrow \Delta$ for each $\mathcal{L}^+$ and each $L \in \bigcup\{SK(G) \mid G \in \mathcal{LG}r(\mathcal{L}^+)\}$.*

By the antecedent of (*), L satisfies every sequent it does not cover. A kernel model is a special case when L is a kernel; it covers all sentences.

Semikernels provide a model for reasoning that explodes from contradictions but not from paradoxes. A contradiction is a formula having no semikernel model. For instance, no semikernel of the graph $\wedge \to \neg A \to A \ldots$ contains its source vertex $A \wedge \neg A$. The apparent coherence of paradoxes, on the other hand, amounts to their possessing semikernel models which, however, cannot be extended to any kernel model evaluating consistently all sentences of the language.

Example 2.8. *Liar K says 'Everything I'm saying is false', $K(Kl)$ with $Kl = \forall\phi(K\phi \to \neg\phi)$, and says only that, $\forall\phi(K\phi \to \phi \doteq Kl)$. $K!Kl$ abbreviates these two sentences. A model must satisfy atoms framed on the drawing: $K(Kl)$ and $\neg K(S)$, for all $S \neq Kl$.*



The framed literals form a semikernel model that has no extension to a kernel: $K(Kl) = \mathbf{1}$ *makes* $\neg K(Kl) = \mathbf{0}$*, while* $\neg K(S) = \mathbf{1}$ *makes* ${}_S\wedge_{K(S)} = \mathbf{0}$*, for* $S \not\doteq Kl$*. The resulting unresolved odd cycle* $\neg Kl - Kl - {}_{Kl}\wedge_{K(Kl)}$ *blocks any evaluation satisfying (2.2).*

While contradictions evaluate simply to false, paradoxes make evaluation of some sentences impossible. The unresolved odd cycle reflects the entailment from K's paradoxical claim to K both lying and not lying: $K!Kl \vDash Kl \wedge \neg Kl$*. This paradox disappears if* $K(X)$ *holds for some true sentence* X*, since then* ${}_X\wedge_{K(X)} = \mathbf{1}$ *makes* $Kl = \mathbf{0}$*.*

Operator K satisfying $K!Kl$ leads to a paradox. This is enabled by the self-referential capacity of LSO. While all sentences of the object-language $\mathcal{L}^-$ receive unique values under every valuation of atoms, certain valuations of operators lead to paradoxes which prevent evaluation of some sentences. In principle, it might happen that language $\mathcal{L}$ itself is inconsistent in this sense, disabling evaluation of some sentences. The existence of kernels in language graphs is a nontrivial claim, demonstrating that this is not the case – every LSO language $\mathcal{L}$, with the rules of classical logic, is consistent, that is, has an interpretation of all sentences not leading to any contradictions.

THEOREM 2.9 ([8, Th.3.4]). *For every LSO language* $\mathcal{L}$*, every graph* $G_M(\mathcal{L})$ *has a kernel.*

A *definitional extension* introduces a new operator O into a language $\mathcal{L}$ by an axiom:

$$\forall\phi, x\big(O(\phi, x) \leftrightarrow \exists\!\!\!/\psi\exists\!\!\!/y(F(\psi, y, \phi', x'))\big), \tag{2.10}$$

where $\exists\!\!\!/$ are any quantifiers and $\exists\!\!\!/\psi\exists\!\!\!/y(F(\psi, y, \phi', x'))$ is an $\mathcal{L}$ formula with the free variables ϕ', x' contained among free ϕ, x of the left side. A sequence of such extensions is also definitional. It is *(model) conservative*, i.e., each kernel model of any original theory can be extended to a kernel model of the extended one. In particular, definitional extension does not introduce any paradoxes.

THEOREM 2.11 ([8, Th.5.6]). *Definitional extension is model-conservative.*

2.1. Reasoning. Sequent system LSO, for reasoning in $\mathcal{L}$ without $\doteq$, extends the classical one with two rules for the s-quantifiers. $\Gamma \vdash \Delta$ denotes the provability of sequent $\Gamma \Rightarrow \Delta$.

(Ax) $\Gamma \vdash \Delta$ – for Γ and Δ sharing some formula (not necessarily atomic)

$(\neg_L)\ \dfrac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$ $\qquad (\neg_R)\ \dfrac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$

$(\wedge_L)\ \dfrac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$ $\qquad (\wedge_R)\ \dfrac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$

$(\forall_L)\ \dfrac{F[x\backslash t], \Gamma \vdash \Delta}{\forall x F(x), \Gamma \vdash \Delta}$ $\qquad (\forall_R)\ \dfrac{\Gamma \vdash \Delta, F[x\backslash y]}{\Gamma \vdash \Delta, \forall x F(x)}$ fresh $y \in o\mathcal{V}$

$(\forall_L^\phi)\ \dfrac{F[\phi\backslash S], \Gamma \vdash \Delta}{\forall\phi F(\phi), \Gamma \vdash \Delta}$ $\qquad (\forall_R^\phi)\ \dfrac{\Gamma \vdash \Delta, F[\phi\backslash\theta]}{\Gamma \vdash \Delta, \forall\phi F(\phi)}$ fresh $\theta \in s\mathcal{V}$

All substitutions in $(\forall)$ rules must be *legal*, i.e., not capture any free variables of the introduced terms/formulas. For instance, substitution of a sentence is always legal and $\forall x\exists\phi(Px \leftrightarrow \phi)$ is derivable – instantiating x with a fresh y, we then (moving bottom-up) instantiate ϕ with Py. In general, $\exists\phi\forall x(Px \leftrightarrow \phi)$ is not derivable, as the substitution $(\forall x(Px \leftrightarrow \phi))[\phi\backslash Px]$ is illegal. Substitution into an equation $(L \doteq R)[\phi\backslash S]$ is legal iff both $L[\phi\backslash S]$ and $R[\phi\backslash S]$ are.

Rules $(\forall_L^\phi)$, $(\forall_R^\phi)$ remind of second-order, but the logic is compact, because quantifiers range only over sentences, not over predicate positions or arbitrary subsets. (Infinite Γ, Δ are allowed, since also proofs using them involve only finite subsets.) In spite of this quantification over its own syntax, LSO is not even a weak second-order logic that quantifies, e.g., only over definable sets. Its closest relative seems rather two-sorted first-order logic.

LSO is sound and complete for the semikernel semantics, but we address this along with the rules for s-equality. Its intended interpretation is fixed by restricting (semi)kernels to the *relevant* ones, interpreting $\doteq$ as (a subset of) the diagonal of $\boldsymbol{S}_M$. Relevant semi/kernels are denoted $SK^{\doteq}/Ker^{\doteq}$.

DEFINITION 2.12. *In language graphs* $G_M(\mathcal{L}^{\doteq})$*, the* relevant *(semi)kernels contain neither negation of any trivial equality,* $S \not\doteq S$*, nor any equality* $S \doteq T$ *for syntactically distinct* $S, T \in \boldsymbol{S}_M$*.*

With this restriction, Theorem 2.9 requires some additional argument. The rest of this section establishes its counterpart, Theorem 2.23, for the intended semantics of $\mathcal{L}^{\doteq}$. The proof uses the extension of LSO to reasoning with equations, $\mathsf{LSO}^{\doteq}$, which relies on an adaptation of the standard decidable unification of formulas, treating quantifiers as operators.

Unification of quantified formulas can be defined in various ways, depending on the strictness in handling the names of bound variables. E.g., in the context of semantic evaluation, $\forall xAx$ and $\forall yAy$ are naturally unifiable, but syntactic equality is a finer relation. $\forall x$ and $\forall y$ are here treated as different 'operator symbols' that do not unify. On the other hand, $\forall xA(x,c)$ and $\forall xA(c,x)$ could be taken as unifiable by the syntactic replacement to $\forall xA(c,c)$. Although unification does not aim at capturing the semantics, it should not violate it by unifying syntactically distinct, not to mention non-equivalent, sentences. To facilitate this, we use a new syntactic category $\dot{\mathcal{V}}$, with a constant $\dot{v}$ for each $v \in \mathcal{V} = s\mathcal{V} \cup o\mathcal{V}$ bound in the actual formula, unifiable only with itself, not even with variables (which would amount to a substitution of/for a bound variable). These auxiliary symbols $\dot{\mathcal{V}}$ serve merely the construction of unifiers during reasoning and occur only there in equations.

Definition 2.13. *Unification, with* $v \in \mathcal{V}$, $A, B \in \boldsymbol{F}_{\mathcal{V},\dot{\mathcal{V}}} \cup \boldsymbol{T}_{o\mathcal{V},o\dot{\mathcal{V}}}$*:*

1.	$E \cup \{v \doteq A\}$	$\rightsquigarrow$	$E[v\backslash A] \cup \{v \doteq A\}$	– *only legal substitutions* $\_[v\backslash A]$
2.	$E \cup \{P(A_i) \doteq P(B_i)\}$	$\rightsquigarrow$	$E \cup \{A_i \doteq B_i\}$	– $P \in \boldsymbol{Pr} \cup \boldsymbol{Op} \cup \{\wedge, \neg\}$
3.	$E \cup \{\forall vA \doteq \forall vB\}$	$\rightsquigarrow$	$E \cup \{A[v\backslash\dot{v}] \doteq B[v\backslash\dot{v}]\}$	– $\dot{v} \in \dot{\mathcal{V}}$ *fresh in* A, B *and* E
4.	$E \cup \{A \doteq A\}$	$\rightsquigarrow$	E	– *remove trivial equations*
5*a*.	$E \cup \{P(...) \doteq Q(...)\}$	$\rightsquigarrow$	*NO*	– *distinct* $P, Q \in \boldsymbol{Pr} \cup \boldsymbol{Op} \cup \{\wedge, \neg, \forall\_\}$
5*b*.	$E \cup \{v \doteq A(v)\}$	$\rightsquigarrow$	*NO*	– *occurs-check failure,* v *in* $A(v)$
5*c*.	$E \cup \{B \doteq A(\dot{v})\}$	$\rightsquigarrow$	*NO*	– $\dot{v}$ *in* $A(\dot{v})$*, but not in* B

A unifier is a set of equations, each having on one side a variable not occurring elsewhere, and NO marks that it does not exist.

In point 1, A is only legally substituted into the sides of $\doteq$. Unification fails in the following examples because it would require an illegal substitution (shown to the right):

$\forall xAxa \doteq \forall xAax, x \doteq a \overset{3}{\rightsquigarrow} A\dot{x}a \doteq Aa\dot{x}, x \doteq a \overset{2}{\rightsquigarrow} \dot{x} \doteq a, x \doteq a \overset{5c}{\rightsquigarrow} NO$ – $\forall xAxa[x\backslash a]$

$\forall\phi.\psi \doteq \forall\phi.B\phi \overset{3}{\rightsquigarrow} \psi \doteq B\dot{\phi} \overset{5c}{\rightsquigarrow} NO$ – $(\forall\phi.\psi)[\psi\backslash B\phi]$

$\forall x\phi \doteq \forall x\forall yPxyz, \forall x\phi \doteq \forall x\forall yPxya \overset{3}{\rightsquigarrow} \phi \doteq \forall yP\dot{x}yz, \phi \doteq \forall yP\dot{x}ya \overset{5c}{\rightsquigarrow} NO$ – $\forall x\phi[\phi\backslash\forall yPxya]$

Unification of quantifiers requires the same bound variable name, introducing the same fresh $\dot{v}$. The result is unchanged if $\dot{v}$ is only *locally fresh* in the equation, possibly occurring in others.

Fact 2.14. *A system* E *of s-equations is unifiable by* σ *choosing a locally fresh* $\dot{v}$ *in step 3 iff it is unifiable by a* σ' *choosing there always a fresh* $\dot{v}$*.*

Proof. ($\Leftarrow$) A successful unification ends with $\dot{v} \doteq \dot{v}$ for every $\dot{v} \in \dot{\mathcal{V}}$, so if σ' unifies all equations, then also σ does that, reusing only some $\dot{v}_i$ (distinct equations $\dot{v}_i \doteq \dot{v}_i$ and $\dot{v}_j \doteq \dot{v}_j$ in σ' become one equation $\dot{v} \doteq \dot{v}$ in σ).

($\Rightarrow$) If no σ' unifies E but some σ does, then some expression with $\dot{v}_1$ from equation E_1 must be reused, i.e., substituted into another equation E_2. Say, E_2 is $L(v_2) \doteq R$ and the unification requires substituting for v_2 an expression with $\dot{v}_1$, say, from E_1 which must have the form $v_2 \doteq Q\dot{v}_1$. But unification fails with this last equation by point 5c of Definition 2.13. □

Reasoning system $\mathsf{LSO}^{\doteq}$ extends LSO with axiom schema (uniAx) and the following five rules (each substitution in (rpL) and (rpR) must be legal):

(uniAx) $\Gamma, A \doteq B \vdash \Delta$ – for non-unifiable A, B

(ref) $$\frac{A \doteq A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

(unif) $$\frac{A_1 \doteq B_1, ...A_i \doteq B_i, \Gamma \vdash \Delta}{O(A_1...A_i) \doteq O(B_1...B_i), \Gamma \vdash \Delta}$$ $O \in \boldsymbol{Pr} \cup \boldsymbol{Op} \cup \{\wedge, \neg\}$

(rpL) $$\frac{F[\phi\backslash A], A \doteq B, \Gamma \vdash \Delta}{F[\phi\backslash B], A \doteq B, \Gamma \vdash \Delta}$$

(unif$^{\forall}$) $$\frac{A[v\backslash\dot{v}] \doteq B[v\backslash\dot{v}], \Gamma \vdash \Delta}{\forall vA \doteq \forall vB, \Gamma \vdash \Delta}$$ fresh $\dot{v}$

(rpR) $$\frac{A \doteq B, \Gamma \vdash \Delta, F[\phi\backslash A]}{A \doteq B, \Gamma \vdash \Delta, F[\phi\backslash B]}$$ ϕ not in $\doteq$

(i) Rules (rpL)/(rpR) replace one side of $A \doteq B$ by another, on the left/right of $\vdash$, provided that both substitutions $F[\phi\backslash A]$ and $F[\phi\backslash B]$ are legal. Otherwise, the rules become unsound, as the following example shows (marking illegal substitution of the underlined subformula by (rpR)–).

$$\dfrac{A(x,a) \doteq A(a,x), \forall x A(x,a) \vdash \forall x \underline{A(x,a)}}{A(x,a) \doteq A(a,x), \forall x A(x,a) \vdash \forall x A(a,x)}\ (\text{rpR})–$$

Since $x \doteq a$ unifies $A(x,a) \doteq A(a,x)$, the unsound conclusion amounts to $\forall x A(x,a)$ entailing $\forall x A(a,x)$. Also substitutions into equations must be legal, as the following illustrates:

$$\dfrac{\dfrac{\dfrac{\forall x A(a,x) \doteq \forall x A(x,a), A(x,a) \doteq A(a,x), \underline{\forall x A(a,x)} \vdash \forall x A(a,x)}{\forall x A(a,x) \doteq \forall x \underline{A(x,a)}, A(x,a) \doteq A(a,x), \forall x A(x,a) \vdash \forall x A(a,x)}\ (\text{rpL})}{\forall x A(a,x) \doteq \forall x A(a,x), A(x,a) \doteq A(a,x), \forall x A(x,a) \vdash \forall x A(a,x)}\ (\text{rpL})–}{A(x,a) \doteq A(a,x), \forall x A(x,a) \vdash \forall x A(a,x)}\ (\text{ref})$$

(ii) Rule (rpR) is unnecessary for reasoning with mere equalities and its side condition prevents substitutions for ϕ occurring in any equation, to simplify the treatment of s-equality. It is needed for connecting nominal and sentential occurrences across the sequent, e.g.:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, \theta \doteq (A \vee \neg A) \vdash A}{\theta \doteq (A \vee \neg A) \vdash A, \neg A}\ (\neg_R)}{\theta \doteq (A \vee \neg A) \vdash A \vee \neg A}\ (\vee_R)}{\theta \doteq (A \vee \neg A) \vdash \theta}\ (\text{rpR})}{\vdash \theta \doteq ((A \vee \neg A) \to \theta)}\ (\to_R)}{\vdash \forall\phi(\phi \doteq ((A \vee \neg A) \to \phi))}\ (\forall^{\phi}_{R})$$

(iii) Non-unifiability in (uniAx) can be restricted to cases 5 in Definition 2.13, as (unif), (unif$^{\forall}$) and (rpL) perform other unification steps. (Unif) discharges an operator/connective O leaving the (pointwise) equalities between its arguments $A_i \doteq B_i$.[3] (unif$^{\forall}$) handles the case not covered by (unif), when both sides of $\doteq$ start with $\forall v$.

The following example applies reasoning with s-equality (at one place) to the liar from Example 2.8, signalling also the non-explosive character of reasoning with the semikernel semantics.

Example 2.15. *Recall that $K!Kl$ abbreviates $Kl = \forall\phi(K\phi \to \neg\phi)$ and $\forall\phi(K\phi \to \phi \doteq Kl)$. The unresolvable odd cycle is now reflected by the provability of K always lying and not always lying. (Sequents over sets give implicit (C)ontraction.)*

$$(1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{Kl, K(Kl) \vdash K(Kl) \qquad \dfrac{Kl, K(Kl) \vdash Kl}{Kl, K(Kl), \neg Kl \vdash}}{Kl, K(Kl), K(Kl) \to \neg Kl \vdash}}{Kl, K(Kl), \forall\phi(K\phi \to \neg\phi) \vdash}\ (\forall^{\phi}_{L})}{K(Kl), \forall\phi(K\phi \to \neg\phi) \vdash}\ (C)}{\dfrac{K(Kl), Kl \vdash}{K(Kl) \vdash \neg Kl}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,(1)}{K(Kl), \theta \doteq Kl, K\theta, Kl \vdash}}{K(Kl), \theta \doteq Kl, K\theta, \theta \vdash}\ (rpL)}{K(Kl), \theta \doteq Kl, K\theta \vdash \neg\theta} \qquad K(Kl), K\theta \vdash \neg\theta, K\theta}{K(Kl), K\theta \to \theta \doteq Kl, K\theta \vdash \neg\theta}}{\dfrac{\dfrac{K(Kl), \forall\phi(K\phi \to \phi \doteq Kl) \vdash K\theta \to \neg\theta}{K(Kl), \forall\phi(K\phi \to \phi \doteq Kl) \vdash \forall\phi(K\phi \to \neg\phi)}\ (\forall^{\phi}_{R})\ \textit{fresh}\ \theta}{K(Kl), \forall\phi(K\phi \to \phi \doteq Kl) \vdash Kl}}\ (\forall^{\phi}_{L})\ \textit{legal}\ (K\phi \to \phi \doteq Kl)[\phi\backslash\theta]$$

As in natural reasoning, the liar lies and does not lie, but not much else follows. The semikernel model from Example 2.8 can be extended to one satisfying all $\mathcal{L}^-$ tautologies and no $\mathcal{L}^-$ contradictions. None of the latter is derivable from $K!Kl$, which itself is not a contradiction. Unlike in the

[3] O can be a predicate symbol with $A, B \in \boldsymbol{T}_{o\mathcal{V},o\dot{\mathcal{V}}}$, extending $\doteq$ and the unification in $\mathsf{LSO}^{\doteq}$ to terms. This unproblematic extension is not elaborated, as it is taken care of by the unification proper in Definition 2.13.

kernel semantics, the fact $K!Kl \vDash Kl \wedge \neg Kl$ *no longer signifies the lack of models, but only that no semikernel containing* $K!Kl$ *covers* $Kl \wedge \neg Kl$*, hence none covers* Kl *or* $\neg Kl$*.*

Since $K!Kl \nvdash A \wedge \neg A$ *(e.g., for an* $\mathcal{L}^-$ *sentence* A*), while* $Kl \wedge \neg Kl \vdash A \wedge \neg A$*, (cut) is not admissible outside the object-language. Adding its unrestricted version narrows the semikernel semantics to kernels, turning the logic exploding only from contradictions, but not paradoxes, into one where also paradoxes entail everything becoming plain contradictions, [8].*

2.2. Consistency of $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$. $\mathsf{LSO}^{\doteq}$ is sound and complete for the relevant semikernel semantics, Definitions 2.7/2.12. The proofs have some novel elements due to the novelty of the language and semantics but, generally, follow the standard route and are given in the appendix.

Fact 2.16 (5.1). *If* $\Gamma \vdash \Delta$ *in* $\mathsf{LSO}^{\doteq}$ *then* $\Gamma \vDash \Delta$*.*

Fact 2.17 (5.3). *If* $\Gamma \nvdash \Delta$ *in* $\mathsf{LSO}^{\doteq}$ *then there is a graph* $G \in \mathcal{LG}r(\mathcal{L}^+)$*, for some* $\mathcal{L}^+ \supseteq \mathcal{L}$*, with an* $L \in SK^{\doteq}(G)$*, such that (i)* $\Gamma \subseteq L$ *and (ii)* $\Delta \subseteq \boldsymbol{E}^-(L)$*.*

The last fact is used frequently, ensuring covering semikernels for unprovable sequents. Unlike in the usual cases, soundness does not establish the consistency of $\mathsf{LSO}^{\doteq}$. It admits, at least in principle, both $\varnothing \vdash A$ and $\varnothing \vdash \neg A$, for a sentence A that is not covered by any semikernel. If such an A exists then no language graph possesses a kernel. Establishing consistency of $\mathsf{LSO}^{\doteq}$, this section shows also that kernels with the intended interpretation of $\doteq$ do exist.

A theory (set of $\mathcal{L}$ formulas) Γ is *p-consistent* if $\Gamma \nvdash A$ or $\Gamma \nvdash \neg A$, for each $\mathcal{L}$ sentence A, and *complete* if it is p-consistent and $\Gamma \vdash A$ or $\Gamma \vdash \neg A$ for each $\mathcal{L}$ sentence A.[4] P-consistency in $\mathsf{LSO}^{\doteq}$ has finite character: a proof of any A from possibly infinite Γ is finite. It is easy to see that $\mathsf{LSO}^{\doteq}$ does not prove the empty sequent, $\varnothing \nvdash \varnothing$, but this does not suffice for its p-consistency (that is, p-consistency of the empty theory $\varnothing$). As noted in the previous paragraph, even soundness for semikernel semantics does not suffice, but it underlies the following proof of p-consistency, which requires also the existence of covering semikernels. First we register a fact used in the proof. ($\mathcal{L}_i^+$ denotes an extension of language $\mathcal{L}$ with additional constants, Definition 2.7.)

Lemma 2.18. *For* $A, B \in \boldsymbol{S}$*, if* $\varnothing \vdash A$ *and* $\varnothing \vdash B$ *and some* $G_A \in \mathcal{LG}r(\mathcal{L}_1^+)$ *has a (relevant) semikernel* L_A *containing* A *and some* $G_B \in \mathcal{LG}r(\mathcal{L}_2^+)$ *a (relevant) semikernel* L_B *containing* B*, then some* $G \in \mathcal{LG}r(\mathcal{L}_3^+)$ *has a (relevant) semikernel containing* A *and* B*.*

Proof. Let $\mathbf{Sk}$ be all (relevant) semikernels of all $\mathcal{LG}r(\mathcal{L}^+)$, for all $\mathcal{L}^+ \supseteq \mathcal{L}$, and $Th(L) = \{T \in \boldsymbol{S} \mid L \vdash T\}$ for $L \in \mathbf{Sk}$. Then $Th(\mathbf{Sk}) = \bigcap_{L \in \mathbf{Sk}} Th(L) \neq \varnothing$, containing, e.g., A, B and all tautologies. $Th(L_B) \nvdash \neg B$ by Fact 2.16 and $B \in L_B$, hence $Th(\mathbf{Sk}) \nvdash \neg B$. Since $A \in Th(\mathbf{Sk})$, Fact 2.17 gives a (relevant) $L \in SK(G)$, for some $G \in \mathcal{LG}r(\mathcal{L}^+)$, with $A \in L$ and $\neg B \in \boldsymbol{E}^-(L)$, i.e., also $B \in L$. □

Let $\mathbf{C}$ range over all, possibly empty, sets of trivial equations $F \doteq F$. For any Γ, Δ and any such $\mathbf{C}$, if $\Gamma \vdash \Delta$ then $\mathbf{C}, \Gamma \vdash \Delta$ (by admissible weakening) and vice versa by (ref).

Lemma 2.19. *For every formula* A *and sets* $\mathbf{C}_1, \mathbf{C}_2$ *of trivial equations over* $\mathcal{L}$*: if* $\mathbf{C}_1 \vdash A$ *then (a)* $A, \mathbf{C}_2 \nvdash \varnothing$ *and (b)* A *is covered by a (relevant) semikernel in some* $G \in \mathcal{LG}r(\mathcal{L}^+)$*.*

Proof. **(i)** By Definition 2.1 of language graphs all atoms, in particular, equations and inequations have covering semikernels at the atomic 2-cycles. The relevant semikernels for equations, covering both dual literals, are all $S \doteq S$ and $S \not\doteq T$ for syntactically distinct S, T. If $\mathbf{C}_1 \vdash A$ for an equation A, then soundness of $\mathsf{LSO}^{\doteq}$ for equations gives validity of A and $A \nvdash \varnothing$, hence $A, \mathbf{C}_2 \nvdash \varnothing$ by (ref). The rest of the proof shows the claim for A that is not an equation. Fact 2.17 provides a semikernel containing A whenever $A, \mathbf{C}_2 \nvdash \varnothing$, hence (a) implies (b). We show $\mathbf{C}_1 \nvdash A$ or $A, \mathbf{C}_2 \nvdash \varnothing$ supposing the contrary and proceeding by induction on the length of a shortest proof of the first sequent $\mathbf{C}_1 \vdash A$ (over all $\mathbf{C}_1$) and, secondarily, of a shortest proof of the second sequent $A, \mathbf{C}_2 \vdash \varnothing$.

(ii) Axiom (uniAx) of the required format gives the second sequent $B \doteq C, \mathbf{C}_2 \vdash \varnothing$ with non-unifiable B, C. Every semikernel covering (any instance of) $B \doteq C$ contains then (the respective

[4] P-consistency entails the semantic consistency – the existence of kernels in language graphs for $\mathcal{L}^{\doteq}$. This, however, is the main Theorem 2.23 of this section and until it is shown distinguishing the two may be helpful.

instance of) $B \not\doteq C$. By soundness for equations, Fact 2.16, $\varnothing \not\vdash B \doteq C$, hence also $\mathbf{C}_1 \not\vdash B \doteq C$ by (ref). If the first sequent follows by (Ax) $B \doteq C \vdash B \doteq C$, then it actually has the form $A \doteq A \vdash A \doteq A$, and soundness for equations ensures $A \doteq A \not\vdash \varnothing$.

(iii) For proofs longer than 1, we consider the last rule in the supposed proof of $\mathbf{C}_1 \vdash A$. If the second sequent follows by (ref) $\dfrac{C \doteq C, \mathbf{C}_2, A \vdash \varnothing}{\mathbf{C}_2, A \vdash \varnothing}$ then IH on its premise gives the claim, and such cases are not mentioned.

Otherwise, if the principal formula of the last step is in $\mathbf{C}_1$, it can result from an application of either (unif), (unif$^\forall$) or (rpL). The former two have then only trivial equations on the left of the premise, and IH can be applied to it. The same is done with (rpL) if it uses only trivial equations, but it may also use a nontrivial one $B \doteq C$:

$$\text{(rpL)}\ \frac{B \doteq C, P[B] \doteq P[C], \mathbf{C}_2' \vdash \varnothing}{\underbrace{B \doteq C}_{A}, \underbrace{P[B] \doteq P[B], \mathbf{C}_2'}_{\mathbf{C}_2} \vdash \varnothing}.$$

This can occur only for the second sequent and was addressed in **(i)**. ($B \doteq C, \mathbf{C}_2 \vdash \varnothing$ implies, by equational soundness, $\not\vdash B \doteq C$.)

(iv) Consequently, we consider only the cases when A is principal in both sequents.

($\neg$) If $\dfrac{A, \mathbf{C}_1 \vdash \varnothing}{\mathbf{C}_1 \vdash \neg A}$ and $\dfrac{\mathbf{C}_2 \vdash A}{\neg A, \mathbf{C}_2 \vdash \varnothing}$ then one of the two assumptions fails by IH, and some semikernel L covers A. If $A \in L$ then L covers $\neg A$, while if $A \in \boldsymbol{E}^-(L)$, then adding $\neg A$ to L yields a semikernel covering $\neg A$.

($\wedge$) Suppose that both

$$(\wedge_R)\ \frac{\mathbf{C}_1 \vdash A \quad \mathbf{C}_1 \vdash B}{\mathbf{C}_1 \vdash A \wedge B} \text{ and } \frac{A, B, \mathbf{C}_2 \vdash \varnothing}{A \wedge B, \mathbf{C}_2 \vdash \varnothing}\ (\wedge_L).$$

By IH, both $A \not\vdash \varnothing$ and $B \not\vdash \varnothing$. Fact 2.17 gives semikernels L_A containing A, and L_B containing B. Lemma 2.18 yields then a semikernel L containing both A and B, in some $G \in \mathcal{LG}r(\mathcal{L}^+), \mathcal{L}^+ \supseteq \mathcal{L}$. Since $\vdash A \wedge B$, every semikernel covering $A \wedge B$ contains it. If $A \wedge B \vdash \varnothing$ then, by soundness, no semikernel covering $A \wedge B$ contains it, contradicting the previous sentence and the existence of L.

($\forall$) Suppose that $(\forall^\phi_R)\dfrac{\mathbf{C}_1 \vdash F\theta}{\mathbf{C}_1 \vdash \forall\phi F\phi}$, with fresh $\theta \in \Theta$, and $(\forall^\phi_L)\dfrac{F(S), \mathbf{C}_2 \vdash \varnothing}{\forall\phi F\phi, \mathbf{C}_2 \vdash \varnothing}$.

The proof $\mathbf{C}_1 \vdash F\theta$ is shorter than the proof $\mathbf{C}_1 \vdash \forall\phi F\phi$. Since (Ax) does not require atoms, also (a copy of) the former with S substituting θ, i.e., $\mathbf{C}_1 \vdash F(S)$, is shorter than the latter. If also $F(S), \mathbf{C}_2 \vdash \varnothing$ then this contradicts IH. (The case of $(\forall_R)$-$(\forall_L)$ is analogous.)

(rpR) $\dfrac{\mathbf{C}_1 \vdash F(B)}{\mathbf{C}_1 \vdash F(C)}$ – $\mathbf{C}_1$ has only trivial equations, so $F(C) \doteq F(B)$, and IH gives the claim. □

The relevant special case gives p-consistency of $\mathsf{LSO}^{\doteq}$: if $\varnothing \vdash A$ then $A \not\vdash \varnothing$, hence also $\varnothing \not\vdash \neg A$. This leads to the main claim of the section: the semantic consistency of $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$, witnessed by the existence of kernels in all language graphs of $\mathcal{L}$. It follows from the following two facts: Lindenbaum's lemma for arbitrary $\mathcal{L}$, and a complete theory determining a kernel model.

FACT 2.20. *Every p-consistent theory* Γ*, over any* $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$*, has a complete extension* $\overline{\Gamma}$*.*

PROOF. First, Γ is p-consistent iff it is *dp-consistent*, namely,

for every finite $F \subset \boldsymbol{S}$, $\Gamma \not\vdash \bigvee_{S\in F}(S \wedge \neg S)$,

since for each $S \in \boldsymbol{S} : S \wedge \neg S \vdash \varnothing$, hence $\Gamma, \bigvee_{S\in F}(S \wedge \neg S) \vdash \varnothing$, so by p-consistency $\Gamma \not\vdash \bigvee_{S\in F}(S \wedge \neg S)$.

Enumerate all sentences $S_1, S_2, ...$ of $\mathcal{L}$ (for uncountable $\mathcal{L}$, use AC to well-order the sentences and take unions in the limits), let $\Gamma_0 = \Gamma$,

$$\Gamma_{i+1} = \begin{cases} \Gamma_i, S_i & \text{if } \Gamma_i, S_i \not\vdash \bigvee_{B\in F}(B \wedge \neg B), \text{ for every finite } F \subset \boldsymbol{S} \\ \Gamma_i, \neg S_i & \text{if } \Gamma_i, S_i \vdash \bigvee_{B\in F}(B \wedge \neg B), \text{ for some finite } F \subset \boldsymbol{S} \end{cases} \quad \text{and} \quad \overline{\Gamma} = \bigcup_{i\in\omega} \Gamma_i.$$

Γ_0 is dp-consistent by Lemma 2.19, and suppose Γ_{i+1} is the first that is not, i.e., $\Gamma_i, S_i \vdash BB$ and $\Gamma_i, \neg S_i \vdash CC$, for some finite $F, G \subset \boldsymbol{S}$ and $BB = \bigvee_{B\in F}(B \wedge \neg B)$ and $CC = \bigvee_{C\in G}(C \wedge \neg C)$. Then $\Gamma_i \vdash \neg S_i, BB$ and $\Gamma_i \vdash S_i, CC$, hence $\Gamma_i \vdash (S_i \wedge \neg S_i), BB, CC$, contradicting dp-consistency of Γ_i.

Since each Γ_i is (d)p-consistent, each finite subset of $\overline{\Gamma}$ is p-consistent, and so is $\overline{\Gamma}$. (This gives also the limit cases for uncountable $\mathcal{L}$.) □

For any $\mathcal{L}$ and domain M, set of provable $\mathcal{L}_M$ sentences is p-consistent by Lemma 2.19. It has thus a complete extension which, in turn, determines a kernel model in $G_M(\mathcal{L}_M) = G_M(\mathcal{L})$.

FACT 2.21. *For any language $\mathcal{L}$, domain M and $G_M \in \mathcal{LG}r(\mathcal{L})$, any complete theory Γ over $\mathcal{L}_M$ determines a kernel $K = \{S \in \boldsymbol{S}_M \mid \Gamma \vdash S\} \in Ker(G_M)$. If $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$ then $K \in Ker^{\doteq}(G_M)$.*

PROOF. Kernel condition (2.2), $F \in K \Leftrightarrow \boldsymbol{E}(F) \cap K = \varnothing$, is verified for each kind of vertex F.

1. For $F \in \boldsymbol{A}_M$, exactly one of $F \in K$ or $\neg F \in K$, so $F \in K \Leftrightarrow \boldsymbol{E}(F) = \neg F \notin K$.
2. For $\neg F$, completeness of Γ gives $\neg F \in K \Leftrightarrow \Gamma \vdash \neg F \Leftrightarrow \Gamma \nvdash F \Leftrightarrow \boldsymbol{E}(\neg F) = F \notin K$.
3. For a conjunction $(A \wedge B) \in K \Leftrightarrow \Gamma \vdash A \wedge B \Leftrightarrow \Gamma \vdash A \;\&\; \Gamma \vdash B \Leftrightarrow \{A, B\} \subset K$. Completeness of Γ yields then the last equality: $(A \wedge B) \in K \Leftrightarrow \boldsymbol{E}(A \wedge B) = \{\neg A, \neg B\} \cap K = \varnothing$.
4. $\forall x F x \in K \Leftrightarrow \Gamma \vdash \forall x F x \Leftrightarrow \Gamma \vdash F(y)$ for a fresh $y \Leftrightarrow \Gamma \vdash F(t)$ for all $t \in \boldsymbol{T}_M \Leftrightarrow \{F(t) \mid t \in \boldsymbol{T}_M\} \subseteq K$. Since $F(t) \in K$ for all t, point 2 gives $\varnothing = K \cap \{\neg F(t) \mid t \in \boldsymbol{T}_M\} = K \cap \boldsymbol{E}(\forall x F x)$. If $\forall x F x \notin K$ then $\neg \forall x F x \in K$ by completeness of Γ, so $\Gamma, \forall x F x \vdash \varnothing$, hence $\Gamma, F(t) \vdash \varnothing$, for some $t \in \boldsymbol{T}_M$ and $\Gamma \vdash \neg F(t)$, i.e., $\neg F(t) \in K$ and thus $\boldsymbol{E}(\forall x F x) \cap K \neq \varnothing$.
5. $\forall \phi F \phi \in K \Leftrightarrow \Gamma \vdash \forall \phi F \phi \Leftrightarrow \Gamma \vdash F\theta$, for a fresh $\theta \in \Theta \Leftrightarrow \Gamma \vdash F(S)$ for all $S \in \boldsymbol{S}_M \Leftrightarrow \{F(S) \mid S \in \boldsymbol{S}_M\} \subseteq K$ – hence $K \cap \boldsymbol{E}(\forall \phi F \phi) = K \cap \{\neg F(S) \mid S \in \boldsymbol{S}_M\} = \varnothing$ by point 2.
   $\forall \phi F \phi \notin K \Leftrightarrow \Gamma \nvdash \forall \phi F \phi \Leftrightarrow \Gamma \vdash \neg \forall \phi F \phi \Leftrightarrow \Gamma, \forall \phi F \phi \vdash \varnothing \Rightarrow \Gamma, F(S) \vdash \varnothing$ for some $S \in \boldsymbol{S}_M$. Then also $\Gamma \vdash \neg F(S)$ so $\neg F(S) \in K$. But $\neg F(S) \in \boldsymbol{E}(\forall \phi F \phi)$, so $K \cap \boldsymbol{E}(\forall \phi F \phi) \neq \varnothing$.

For $\mathcal{L}_M \supseteq \mathcal{L}^{\doteq}_M$, provability of trivial equations and negations of false ones gives $K \in Ker^{\doteq}(G_M)$. □

As a simple consequence of Facts 2.20 and 2.21 we obtain the following.

COROLLARY 2.22. *A p-consistent Γ has a (relevant) kernel model, over any domain.*

In particular, the empty theory has kernel models since it is p-consistent by Lemma 2.19. In other words, the extension of any classical FOL/HOL language with s-quantifiers and $\doteq$ is consistent – its graphs possess kernels, no contradiction is provable, and no paradoxes arise. This unsurprising theorem, missing in earlier presentations of LSO, is crucial for the next section.

THEOREM 2.23. *Each language graph $G_M(\mathcal{L})$, for any $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$, has a relevant kernel.*

§3. Truth definition. Any given language $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$ is expanded to $\mathcal{L}^{\mathsf{E}} = \mathcal{L} \cup \{\mathsf{E}\}$ with operator E axiomatised by theory Ed to represent the edge relation in the language graphs. Truth is defined in $\mathcal{L}^{\mathsf{E}}$ by an axiom capturing the kernel condition (2.3). To facilitate comparison with AST, the truth operator T is introduced by a definitional extension abbreviating this last step, and yielding theory Tr. For each domain M, we thus obtain three language graphs corresponding to $\mathcal{L} \subset \mathcal{L}^{\mathsf{E}} \subset \mathcal{L}^{\mathsf{T}}$:

$G_M = (\boldsymbol{S}_M, \boldsymbol{E}) \subset G^{\mathsf{E}}_M = (\boldsymbol{S}^{\mathsf{E}}_M, \boldsymbol{E}^{\mathsf{E}}) \subset G^{\mathsf{T}}_M = (\boldsymbol{S}^{\mathsf{T}}_M, \boldsymbol{E}^{\mathsf{T}})$,

with inclusions marking the induced subgraph relations. The essential step is the axiomatisation of E which reflects Definition 2.1. (The unary arguments stand for any arities.)

DEFINITION 3.1. *Theory Ed in $\mathcal{L}^{\mathsf{E}}$ has axiom schemas 1, 4 and axioms 2, 3:*

1*a*.	$\forall x \forall \psi :$	$\mathsf{E}(Px, \psi)$	$\leftrightarrow$	$\psi \doteq \neg Px$	– for each $P \in \boldsymbol{Pr}$
1*b*.	$\forall \phi, \psi :$	$\mathsf{E}(O\phi, \psi)$	$\leftrightarrow$	$\psi \doteq \neg O\phi$	– for each $O \in \boldsymbol{Op}$
2.	$\forall \alpha, \psi :$	$\mathsf{E}(\neg \alpha, \psi)$	$\leftrightarrow$	$\psi \doteq \alpha$	
3.	$\forall \alpha, \beta, \psi :$	$\mathsf{E}(\alpha \wedge \beta, \psi)$	$\leftrightarrow$	$(\psi \doteq \neg \alpha) \vee (\psi \doteq \neg \beta)$	
4*a*.	$\forall v \forall \psi :$	$\mathsf{E}(\forall x A(v, x), \psi)$	$\leftrightarrow$	$\exists y (\psi \doteq \neg A(v, y))$	– for each $\forall x A(v, x) \in \boldsymbol{F}^{\mathsf{E}}_v$
4*b*.	$\forall v \forall \psi :$	$\mathsf{E}(\forall \phi A(v, \phi), \psi)$	$\leftrightarrow$	$\exists \psi (\psi \doteq \neg A(v, \psi))$	– for each $\forall \phi A(v, \phi) \in \boldsymbol{F}^{\mathsf{E}}_v$

Schemas 4 are also for sentences. Axiom 1b for $\mathsf{E}(\alpha, \beta)$ is $\forall \alpha, \beta, \psi : \mathsf{E}(\mathsf{E}(\alpha, \beta), \psi) \leftrightarrow \psi \doteq \neg \mathsf{E}(\alpha, \beta)$. Operator E represents the edge relation $\boldsymbol{E}^{\mathsf{E}}$, in the following sense.

FACT 3.2. *For any M, if $Ed \subseteq K \in Ker^{\doteq}(G^{\mathsf{E}}_M)$ then $\forall A, B \in \boldsymbol{S}^{\mathsf{E}}_M : \mathsf{E}(A, B) \in K \Leftrightarrow \boldsymbol{E}^{\mathsf{E}}(A, B)$.*

Proof. Each case follows in the same way, applying the corresponding points from Definitions 2.1 and 3.1, using $(A \doteq B) \in K$ for each true closed s-equation $A \doteq B$, since K is relevant. Axioms 1b are given only for E and $\doteq$; ψ is any actual sentence $\boldsymbol{S}^{\mathsf{E}}_M$.

1a. For an atom $Pm, m \in M$: $\boldsymbol{E}^{\mathsf{E}}(Pm,\psi) \overset{2.1.1}{\iff} \psi \doteq \neg Pm \overset{2.12}{\Leftrightarrow} \psi \doteq \neg Pm \in K \overset{3.1.1}{\iff} \mathsf{E}(Pm,\psi) \in K$.

1b. $\boldsymbol{E}^{\mathsf{E}}((A \doteq B),\psi) \overset{2.1.1}{\iff} \psi \doteq \neg(A \doteq B) \overset{2.12}{\iff} (\psi \doteq \neg(A \doteq B)) \in K \overset{3.1.1}{\iff} \mathsf{E}((A \doteq B),\psi) \in K$.

$\boldsymbol{E}^{\mathsf{E}}(\mathsf{E}(A,B),\psi) \overset{2.1.1}{\iff} \psi \doteq \neg \mathsf{E}(A,B) \overset{2.12}{\iff} (\psi \doteq \neg \mathsf{E}(A,B)) \in K \overset{3.1.1}{\iff} \mathsf{E}(\mathsf{E}(A,B),\psi) \in K$.

2. For $\neg A \in \boldsymbol{S}^{\mathsf{E}}_M : \boldsymbol{E}^{\mathsf{E}}(\neg A,\psi) \overset{2.1.2}{\iff} \psi \doteq A \overset{2.12}{\iff} (\psi \doteq A) \in K \overset{3.1.2}{\iff} \mathsf{E}(\neg A,\psi) \in K$.

3. For $A, B \in \boldsymbol{S}^{\mathsf{E}}_M : \boldsymbol{E}^{\mathsf{E}}(A \wedge B,\psi) \overset{2.1.2}{\iff} \psi \doteq \neg A$ or $\psi \doteq \neg B \overset{2.12}{\iff} (\psi \doteq \neg A) \in K$ or $(\psi \doteq \neg B) \in K$ $\overset{3.1.3}{\iff} \mathsf{E}(A \wedge B,\psi) \in K$.

4a. For $n \in M : \boldsymbol{E}^{\mathsf{E}}(\forall x A(n,x),\psi) \overset{2.1.4a}{\iff} \psi \doteq \neg A(n,m)$, for some $m \in M \overset{2.12}{\iff}$ $(\psi \doteq \neg A(n,m)) \in K$, for some $m \in M \overset{3.1.4a}{\iff} \mathsf{E}(\forall x A(n,x),\psi) \in K$.

4b. For each $n \in M : \boldsymbol{E}^{\mathsf{E}}(\forall \phi A(n,\phi),\psi) \overset{2.1.4b}{\iff} \psi \doteq \neg A(n,S)$, for some $S \in \boldsymbol{S}^{\mathsf{E}}_M \overset{2.12}{\iff}$ $(\psi \doteq \neg A(n,S)) \in K$, for some $S \in \boldsymbol{S}^{\mathsf{E}}_M \overset{3.1.4b}{\iff} \mathsf{E}(\forall \phi A(n,\phi),\psi) \in K$. □

The following theorem ensures that kernels containing Ed, assumed in the fact above, actually exist. Its proof shows that Ed is essentially a conservative extension. The form of Ed axioms in Definition 3.1 suggests this but, being schematic in the first argument, does not fully conform to the format (2.10). Hence the qualification "essentially" and the need for an additional argument.

The proof of the theorem relies on the following simple consequence of compactness, which is of general interest providing kernels for limit *interpretations* of an operator O (in any $G_M(\mathcal{L})$), that is, the set of $\mathcal{L}_M$ sentences about which O is true, $\{S \in \boldsymbol{S}_M \mid O(S) = \mathbf{1}\}$.

Lemma 3.3. *Let $G = G_M(\mathcal{L})$, $K \in Ker(G)$, $\mathcal{L}^O = \mathcal{L} \cup \{O\}$, for an operator O, $G^O = G_M(\mathcal{L}^O)$, $\{O_i \subseteq \boldsymbol{S}^O_M \mid i \in I\}$ and $O_J = \bigcup_{i\in J} O_i$, for $J \subseteq I$. If for each $J \in \mathcal{P}^{fin}(I)$, G^O has a kernel $K_J \supset K$ with $K_J \cap \{O(S) \mid S \in \boldsymbol{S}^O_M\} = \{O(S) \mid S \in O_J\}$, then it has a kernel $K_I \supset K$ with $K_I \cap \{O(S) \mid S \in \boldsymbol{S}^O_M\} = \{O(S) \mid S \in O_I\}$.*

Proof. Using $\mathcal{L}_M$ and $\mathcal{L}^O_M$, let $K^- = \{\neg S \in \boldsymbol{S}_M \mid S \notin K\}$, $O^+_J = \{O(S) \mid S \in O_J\} \subset K_J$ and $O^-_J = \{\neg O(S) \mid S \in \boldsymbol{S}^O_M \setminus O_J\}$, for each $K_J \supset K$. If no kernel model of O_I contains K, then $O^+_I \cup O^-_I \cup K \cup K^- \vdash A \wedge \neg A$ for some $A \in \boldsymbol{S}^O_M$, by Corollary 2.22. A finite subset, sufficient to deduce this, is contained in $O^+_J \cup O^-_J \cup K \cup K^-$ for some finite $J \subset I$. This contradicts the assumption, so $O^+_I \cup O^-_I \cup K \cup K^-$ is consistent and, by Corollary 2.22, has a kernel model $K_I \supset K$. □

Theorem 3.4. *Extension with Ed, of any theory Γ over $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$, is model-conservative.*

Proof. For $\mathcal{L}^{\mathsf{E}} = \mathcal{L} \cup \{\mathsf{E}\}$, let U comprise all predicate, operator and connective symbols, and all quantified formulas, including sentences, $\boldsymbol{F}^{\mathsf{E}}_v$ (v abbreviating any free variables). For each $u \in U$, we define operator E_u by axioms reflecting those of Ed from Definition 3.1:

1*a*. $\forall \gamma,\psi : \mathsf{E}_P(\gamma,\psi) \leftrightarrow \exists x\big((\gamma \doteq Px) \wedge (\psi \doteq \neg Px)\big)$ for each $P \in \boldsymbol{Pr}$

1*b*. $\forall \gamma,\psi : \mathsf{E}_O(\gamma,\psi) \leftrightarrow \exists \phi\big((\gamma \doteq O\phi) \wedge (\psi \doteq \neg O\phi)\big)$ for each $O \in \boldsymbol{Op}$

2. $\forall \gamma,\psi : \mathsf{E}_\neg(\gamma,\psi) \leftrightarrow \exists \alpha\big((\gamma \doteq \neg\alpha) \wedge (\psi \doteq \alpha)\big)$

3. $\forall \gamma,\psi : \mathsf{E}_\wedge(\gamma,\psi) \leftrightarrow \exists \alpha,\beta\big((\gamma \doteq \alpha \wedge \beta) \wedge ((\psi \doteq \neg\alpha) \vee (\psi \doteq \neg\beta))\big)$

4*a*. $\forall \gamma,\psi : \mathsf{E}_{\forall x A v x}(\gamma,\psi) \leftrightarrow \exists v\big((\gamma \doteq \forall x A(v,x)) \wedge \exists x(\psi \doteq \neg A(v,x))\big)$ for each $\forall x A(v,x) \in \boldsymbol{F}^{\mathsf{E}}_v$

4*b*. $\forall \gamma,\psi : \mathsf{E}_{\forall \phi A v \phi}(\gamma,\psi) \leftrightarrow \exists v\big(\gamma \doteq \forall \phi A(v,\phi) \wedge \exists \phi(\psi \doteq \neg A(v,\phi))\big)$ for each $\forall \phi A(v,\phi) \in \boldsymbol{F}^{\mathsf{E}}_v$

Each axiom introduces operator E_u by a definitional extension. Viewing its interpretation as a partial interpretation of E, model-conservativeness of Ed follows now using Lemma 3.3.

For each $u \in U$, let R_u denote the right side of the axiom for E_u. For each finite $J \subset \mathcal{P}(U)$, $\mathsf{E}(\gamma,\psi) \leftrightarrow \bigvee_{j\in J} R_j(\gamma,\psi)$ is a definitional extension. By Theorem 2.11, for $G_M \in \mathcal{LG}r(\mathcal{L})$, any $K \in Ker^{\doteq}(G_M)$ (hence any $K \supset \Gamma$) can be extended to a $K_J \in Ker^{\doteq}(G^{\mathsf{E}}_M)$ with $\mathsf{E}(A,B) \in K_J$ iff

$\exists j \in J : \mathsf{R}_j(A,B) \in K$ for all $A, B \in \boldsymbol{S}_M^{\mathsf{E}}$, i.e., with the interpretation of E equal to $\mathsf{E}_J = \bigcup_{c \in J} \mathsf{E}_j$. Lemma 3.3 gives then a kernel $K_U \supset K$ interpreting E as the union of all $\bigcup_{J \in \mathcal{P}^{fin}(U)}(\mathsf{E}_J) = \bigcup_{u \in U} \mathsf{E}_u$, i.e., satisfying Ed. □

Now, by Fact 3.2 the operator E represents the edge relation of graph G_M^{E} in kernels $K \in Ker^{\doteq}(G_M^{\mathsf{E}})$ satisfying $K \vDash Ed$. Together with (2.3), this implies that for every such kernel K and sentence $S \in \boldsymbol{S}_M^{\mathsf{E}}$, $K \vDash S$ iff $K \vDash \forall\psi(\mathsf{E}(S,\psi) \to \neg\psi)$, yielding

$$K \vDash \forall\phi(\phi \leftrightarrow \forall\psi(\mathsf{E}(\phi,\psi) \to \neg\psi)). \tag{3.5}$$

Let (KE) stand for the sentence on the right of $\vDash$. In spite of (3.5), $Ed \nvDash$ (KE), because Definition 3.1 is schematic. For the additional constants, admitted in $\mathcal{L}^+ \supset \mathcal{L}^{\mathsf{E}}$ by Definition 2.7, E may fail to represent the edge relation. All instances of (KE) with $\mathcal{L}^{\mathsf{E}}$ sentences are provable from Ed, but the universal statement (KE) is not, as illustrated by the failure of its implication to the right:

$$\begin{array}{cl} Ed, \underline{\theta, \mathsf{E}(\theta,\beta), \beta} \vdash & \\ \hline Ed, \theta, \mathsf{E}(\theta,\beta) \vdash \neg\beta & \\ \hline Ed, \theta \vdash \mathsf{E}(\theta,\beta) \to \neg\beta & \\ \hline Ed, \theta \vdash \forall\psi(\mathsf{E}(\theta,\psi) \to \neg\psi) & \text{fresh } \beta \in \Theta \\ \hline Ed \vdash \theta \to \forall\psi(\mathsf{E}(\theta,\psi) \to \neg\psi) & \\ \hline Ed \vdash \forall\phi(\phi \to \forall\psi(\mathsf{E}(\phi,\psi) \to \neg\psi)) & \text{fresh } \theta \in \Theta \end{array} \tag{3.6}$$

Axioms Ed are inapplicable since they do not cover the case of $\mathsf{E}(\theta,\beta)$. Syntactic variables θ and β of $\mathcal{L}$ become constants of $\mathcal{L}^+$ in the countermodel, where the underlined formulas are true and atom $\mathsf{E}(\theta,\beta)$ no longer represents the edge relation. Although Definition 3.1 covers all actual cases for $\mathcal{L}^{\mathsf{E}}$, no axioms cater for the possible extensions $\mathcal{L}^+$. A complete reasoning about the intended language $\mathcal{L}^{\mathsf{E}}$ would require a more detailed syntax theory, identifying sentences of $\mathcal{L}^{\mathsf{E}}$, or some form of induction. This is left for future work, as it is not needed for the present purpose.

Definition of truth is obtained by extending Ed with axiom:

$$\text{(KE)} \qquad \forall\phi(\phi \leftrightarrow \forall\psi(\mathsf{E}(\phi,\psi) \to \neg\psi)). \tag{3.7}$$

Its right side gives the truth operator, $\mathsf{T}(\phi) \leftrightarrow \forall\psi(\mathsf{E}(\phi,\psi) \to \neg\psi)$; Tr is Ed with (KE) and this definitional extension (adding also missing axioms to Definition 3.1: 1b for T and 4 for $\boldsymbol{F}_z^{\mathsf{T}} \setminus \boldsymbol{F}_z^{\mathsf{E}}$.) The unrestricted Convention T (also for open formulas) follows trivially:

$$Tr \vDash \forall\phi(\mathsf{T}(\phi) \leftrightarrow \phi). \tag{3.8}$$

Given the conservativity of Ed, Fact 3.2, and (3.5), equally obvious is the conservativity of Tr.

Theorem 3.9. *Extension with Tr, of any theory over $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$, is model-conservative.*

Tr axiomatises truth of $\mathcal{L}^{\mathsf{T}}$ sentences, but doing this over arbitrary $\mathcal{L}^+ \supseteq \mathcal{L}^{\mathsf{T}}$ structures, it says also something about the truth of sentences in $\mathcal{L}^+ \setminus \mathcal{L}^{\mathsf{T}}$. Since E captures the edge relation only for graphs $\mathcal{LG}r(\mathcal{L}^{\mathsf{T}})$, but not necessarily for $\mathcal{LG}r(\mathcal{L}^+) \setminus \mathcal{LG}r(\mathcal{L})$, as indicated by (3.6), we comment briefly the meaning of E in the latter.

The edge relation in any graph reflects the direct dependence of a vertex's value on the values of its out-neighbours. When operator E in K – a kernel of $G^+ \in \mathcal{LG}r(\mathcal{L}^+)$ with $K \vDash Tr$ – does not represent $\boldsymbol{E}^+$ on $\boldsymbol{S}_M^+ \setminus \boldsymbol{S}_M$, it still reflects an indirect dependence on some other vertices. Axiom (KE) restricts namely the interpretations to ones where E acts like $\boldsymbol{E}$ in (2.2). For instance, no axiom of Ed requires that an s-constant $D \in \mathcal{L}^+ \setminus \mathcal{L}$ is E-related only to $\neg D$, as axiom 1b ensures for s-constants of $\mathcal{L}$. In any kernel $K \vDash Tr$, however, (KE) admits as $\mathsf{E}(D) = \{x \in \boldsymbol{S}_M^+ \mid \mathsf{E}(D,x) \in K\}$ only sets of sentences whose simultaneous falsity forces $D = \mathbf{1}$, while truth of any makes $D = \mathbf{0}$. The same holds for any other $S \in \boldsymbol{S}_M^+ \setminus \boldsymbol{S}_M$ not covered by axioms Ed. (KE) generalises thus the direct edge dependence – of the value of any $A \in \boldsymbol{S}_M$ on the values of its out-neighbours $\boldsymbol{E}(A)$ – to a possibly indirect dependence E of the value of $S \in \boldsymbol{S}_M^+ \setminus \boldsymbol{S}_M$ on the values of $\mathsf{E}(S)$: for any valuations α, β satisfying (2.2), if $\alpha(S) \neq \beta(S)$ then $\alpha(\psi) \neq \beta(\psi)$ for some $\psi \in \mathsf{E}(S)$. (Adjusting for the contextual differences, this is a dependence relation from [2].)

Clearly, T is type-free and distributes over logical connectives and quantifiers, satisfying the compositional equivalences of Tarski's truth definition.

Fact 3.10. *In $K \in Ker^{\doteq}(G)$ with $Tr \subseteq K$, T distributes over logical connectives and quantifiers.*

PROOF. All cases follow directly from (3.8), for all formulas $A, B, Ax/A\phi$. The equivalences not marked below by 3.8 follow by definitions of language graph and kernel, 2.1/(2.2).

1. $K \vDash \forall x, \phi(\mathsf{T}(At(x,\phi)) \overset{3.8}{\leftrightarrow} At(x,\phi))$ – for all atoms $At(x,\phi)$
2. $K \vDash \mathsf{T}(\neg A) \leftrightarrow \neg \mathsf{T}(A)$, as $\mathsf{T}(\neg A) \in K \overset{3.8}{\Leftrightarrow} \neg A \in K \Leftrightarrow A \notin K \overset{3.8}{\Leftrightarrow} \mathsf{T}(A) \notin K \Leftrightarrow \neg \mathsf{T}(A) \in K$
3. $K \vDash \mathsf{T}(A \wedge B) \leftrightarrow \mathsf{T}(A) \wedge \mathsf{T}(B)$, since
$$\mathsf{T}(A \wedge B) \in K \overset{3.8}{\Leftrightarrow} A \wedge B \in K \Leftrightarrow A \in K \;\&\; B \in K \overset{3.8}{\Leftrightarrow} \mathsf{T}(A) \in K \;\&\; \mathsf{T}(B) \in K \Leftrightarrow \mathsf{T}(A) \wedge \mathsf{T}(B) \in K.$$
4. Only the s-quantifier case is shown – the other is virtually identical. The two tautologies:
   $\vDash \forall\phi(A\phi \leftrightarrow \mathsf{T}(A\phi)) \to (\forall\phi A\phi \to \forall\phi \mathsf{T}(A\phi))$ and
   $\vDash \forall\phi(A\phi \leftrightarrow \mathsf{T}(A\phi)) \to (\forall\phi \mathsf{T}(A\phi) \to \forall\phi A\phi)$, yield
   $K \vDash \forall\phi \mathsf{T}(A\phi) \leftrightarrow \mathsf{T}(\forall\phi A\phi)$, since $K \vDash \forall\phi(A\phi \leftrightarrow \mathsf{T}(A\phi))$ by Fact 3.8. □

These equivalences, expressing the truth value of a composite sentence in terms of the truth values of its (instantiated) subformulas, reflect the fact that the value of each vertex is determined by the values of its out-neighbours which, transitively, give all (instantiated) subformulas of a composite sentence. However, this holds only locally and is coupled with circular dependencies which permeate the subgraph of the metalanguage. Unlike in the usual well-founded languages, a sentence like $\forall\phi.\phi$ is its own instance, so $\mathsf{T}(\forall\phi.\phi) \Leftrightarrow \forall\phi(\mathsf{T}\phi)$ leads to $\mathsf{T}(\forall\phi.\phi)$ reappearing as one of the instances of the right side. The well-defined value of $\forall\phi.\phi$ makes also $\mathsf{T}(\forall\phi.\phi)$ well-defined, but this does not mean any reduction of the evaluation with the more complex argument, e.g., $\mathsf{T}(\forall\phi.\phi)$, to the evaluation with a simpler one, $\mathsf{T}(\phi)$. Compositionality interacts with circularity and, facing sentential quantifiers, does not always reduce complex expressions to simpler ones.

§4. Some philosophical remarks. Theory Tr axiomatises truth of $\mathcal{L}^{\mathsf{T}}$ sentences in all structures, i.e., validity. For a domain M, the corresponding theory in $\mathcal{L}^{\mathsf{T}}_M$ captures truth of its sentences in $G_M(\mathcal{L}^{\mathsf{T}})$. Truth in a specific FOL structure (M,μ), i.e., a domain M with a valuation μ of $\mathcal{L}^-$ atoms, requires in addition an axiomatisation of μ. This is still relative to the capacity of $\mathcal{L}^-$ to axiomatise M. These related issues pertain not so much to the general notion as to its specific instances, relative to the object-language, which may deserve a separate study.

Truth theory is largely separated from syntax theory in LSO. The syntactic equality of sentences suffices for the definition of truth, while any consistent theory, also of syntax, has a consistent extension with the truth operator by Theorem 3.9. Dispensing with Gödelisation severs also the bonds between truth theory and arithmetic. The truth theory in LSO neither benefits nor suffers from any consequences of arithmetic, like the diagonal lemma that introduces both the power of self-reference and the main limitations of AST truth theories ([8] gives its variant for LSO).

As noted, a more detailed syntax theory could perhaps yield a finite axiomatization of truth, so it would certainly be a desirable refinement. A related topic worth further investigation is the formalisation of metaconcepts like provability or consistency within LSO, and the possibility of carrying out the current proofs inside the system.

An interesting aspect has been set aside in order to focus on the main exposition. Truth defined via (KE) functions within the semikernel semantics and does not require totality. When the liar both lies and does not lie, the concept still applies to sentences unaffected by the paradox leaving, for instance, all tautologies true and all object-level contradictions false. This non-explosiveness (from paradoxes) becomes 'fully classical' explosive logic by adding merely the unrestricted version of (cut). It is natural to ask further questions about the interaction between the truth and this partiality, possibly, introducing an operator that identifies the paradoxical claims.

On a more general note, the emerging view of truth seems to be one of redundancy. The ascription of truth to a statement does not add any content to the statement. The extension with Tr is conservative, while the unrestricted Convention T makes each sentence $\mathsf{T}(S)$ equivalent to – hence intersubstitutable at the sentential positions with – S. (This can easily fail at the nominal positions, e.g., for an operator holding only for sentences starting with T.) Such a neutrality becomes a feature of truth under any definition satisfying the unrestricted Convention T.

Despite that, the proposed definition is quite substantial. It captures the semantic structure as well as the relation that serves as the truth condition. To simplify, it declares a sentence true if and only if every sentence it negates is false. (Semantically, even an atomic sentence negates its own negation.) This signals the element of coherence, or rather holism, that is strongly present. Each pair of sententially quantified sentences has in the graph paths both ways, forming a connected component, where every sentence depends in some way on every other, even on itself. In this sense, truth in the metalanguage is genuinely holistic.

But this holism is only half of the story. The subgraph for the object-language is well-founded. (The 2-cycles of dual literals at its leaves represent only possible valuations of atoms.) It provides only an equivalent representation of the classical semantics, where each valuation of atoms induces unique values to all sentences. Tarski viewed truth at this level as a form of correspondence, though critics complained about its elements missing in his framework. Nevertheless, truth for the object-language sentences, determined by atomic facts, certainly carries some correspondence with them, which contrasts with the holistic features of truth in the metalanguage.

The two perspectives complement each other, just as the two language levels do. Truth has, like science in Quine's formulation, "its double dependence upon language and experience", [6]. The part of the language that depends on the language displays some phenomena not occurring in the other part. The truth value of one sentence may depend on the truth value of another which, in turn, depends on the first. The sheer possibility of evaluating coherently all sentences of the metalanguage becomes uncertain and can be disturbed by paradoxical claims. Nothing similar occurs in the object-language. Truth of its statements, concerned exclusively with the nonlinguistic facts, is founded in the correspondence to such facts. It seems highly satisfactory that these two, apparently contrary views, appear as expressions of one formalised concept.

Finally, the formalisation in the language of its semantics and truth does not simplify the issue of deciding the truth of specific sentences. Defining what truth is does not settle the question what is true. Some metalinguistic conundrums get clarified but, in general, the question remains as problematic as it was before. This may be disappointing for the reductionist hopes of a model replacing truth with some simpler concept that would be easier to ascertain. Such hopes seem driven by the the view of truth at the object-level. It remains incomplete without its holistic counterpart at the metalevel, which clarifies primarily the peculiarities of its self-referential ascriptions. Together, they provide an internal representation of the truth concept in the language and a general picture of which facts or sentences may be involved in the verification of the truth of other sentences. Which specific sentences actually are true depends primarily on the way the world is, but also on the claims we make, especially about our own claims.

§5. Appendix. The system obtained from $\mathsf{LSO}^{\doteq}$ by adding to the premise of each rule the principal formula of its conclusion is equivalent by the admissibility of weakening and contraction. Also, replacing (Ax) by its version requiring the formula shared by both sides of $\Rightarrow$ to be atomic yields an equivalent system, admitting its original version. We show soundness and completeness for the so modified system, Figure 1 below. (The new versions of the rules, with primed names, have unchanged side conditions; all substitutions $F[x\backslash\_]$, $F[\phi\backslash\_]$ are legal). Validity $\Gamma \vDash \Delta$, Definition 2.7, is applied for relevant semikernels, Definition 2.12.

Fact 5.1 (2.16). *If* $\Gamma \vdash \Delta$ *in* $\mathsf{LSO}^{\doteq}$ *then* $\Gamma \vDash \Delta$ *and the rules are semantically invertible.*

Proof. By induction on the length of the proof $\Gamma \vdash \Delta$, we verify that given an $\mathcal{L} \supseteq \mathcal{L}^{\doteq}$, for any $\mathcal{L}^{+} = \mathcal{L} \cup \mathbf{C}$, with an arbitrary set $\mathbf{C}$ of additional s-constants, in each language graph $G = G_M(\mathcal{L}^{+})$, every semikernel L covering the conclusion of a rule satisfies it, assuming this for the rule's premises. Invertibility follows since each premise contains the conclusion.

1. (Ax)' is valid for any valuation α. If a semikernel L covers $\alpha(\Gamma) \cup \alpha(\Delta)$ and contains $\alpha(\Gamma)$, then it obviously contains also $\alpha(\Gamma \cap \Delta)$. (uniAx) is valid for the intended interpretation of $\doteq$, since non-unifiability of $A \doteq B \in \Gamma$ means exactly that no substitution yields identical $\alpha(A)$ and $\alpha(B)$, hence no semikernel satisfying Definition 2.12 contains $\alpha(A) \doteq \alpha(B)$.

2. $(\wedge_R)$'. Assume $\Gamma \vDash \Delta, A_1$ and $\Gamma \vDash \Delta, A_2$, and let semikernel L cover the rule's conclusion, under a valuation α. Assume that $\alpha(\Gamma) \cup \alpha(\Delta) \subseteq \boldsymbol{E}^{-}(L)$ and $\alpha(A_1 \wedge A_2) \in \boldsymbol{E}^{-}[L]$ – otherwise the conclusion is satisfied under α. It follows also if $\alpha(A_1 \wedge A_2) \in L$, so suppose $\alpha(A_1 \wedge A_2) \in \boldsymbol{E}^{-}(L)$.

(Ax)' $\Gamma \vdash \Delta$ – Γ and Δ sharing an atom

$(\neg_L)$' $\dfrac{\neg A, \Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$ $\qquad$ $(\neg_R)$' $\dfrac{A, \Gamma \vdash \Delta, \neg A}{\Gamma \vdash \Delta, \neg A}$

$(\wedge_L)$' $\dfrac{A, B, A \wedge B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$ $\qquad$ $(\wedge_R)$' $\dfrac{\Gamma \vdash \Delta, A_1 \wedge A_2, A_1 \quad \Gamma \vdash \Delta, A_1 \wedge A_2, A_2}{\Gamma \vdash \Delta, A_1 \wedge A_2}$

$(\forall_L)$' $\dfrac{F[x \backslash t], \forall x F(x), \Gamma \vdash \Delta}{\forall x F(x), \Gamma \vdash \Delta}$ $\qquad$ $(\forall_R)$' $\dfrac{\Gamma \vdash \Delta, \forall x F(x), F[x \backslash y]}{\Gamma \vdash \Delta, \forall x F(x)}$ fresh y

$(\forall_L^\phi)$' $\dfrac{F[\phi \backslash S], \forall \phi F(\phi), \Gamma \vdash \Delta}{\forall \phi F(\phi), \Gamma \vdash \Delta}$ $\qquad$ $(\forall_R^\phi)$' $\dfrac{\Gamma \vdash \Delta, \forall \phi F(\phi), F[\phi \backslash \theta]}{\Gamma \vdash \Delta, \forall \phi F(\phi)}$ fresh θ

(uniAx) $\Gamma, A \doteq B \vdash \Delta$ – non-unifiable A, B $\qquad$ (ref) $\dfrac{A \doteq A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$

(unif)' $\dfrac{A \doteq B, O(A) \doteq O(B), \Gamma \vdash \Delta}{O(A) \doteq O(B), \Gamma \vdash \Delta}$ $\qquad$ (rpL)' $\dfrac{F[\phi \backslash A], F[\phi \backslash B], A \doteq B, \Gamma \vdash \Delta}{F[\phi \backslash B], A \doteq B, \Gamma \vdash \Delta}$

$(\text{unif}^\forall)$' $\dfrac{A[v \backslash \dot{v}] \doteq B[v \backslash \dot{v}], \forall v A \doteq \forall v B, \Gamma \vdash \Delta}{\forall v A \doteq \forall v B, \Gamma \vdash \Delta}$ fresh $\dot{v}$ $\qquad$ (rpR)' $\dfrac{A \doteq B, \Gamma \vdash \Delta, F[\phi \backslash B], F[\phi \backslash A]}{A \doteq B, \Gamma \vdash \Delta, F[\phi \backslash B]}$ ϕ not in $\doteq$

FIGURE 1. Reasoning system LSO' (all substitutions are legal).

Since $\boldsymbol{E}(\alpha(A_1 \wedge A_2)) = \{\neg\alpha(A_1), \neg\alpha(A_2)\}$ so, for some $i \in \{1, 2\}$, $\neg\alpha(A_i) \in L$, and then $\alpha(A_i) \in \boldsymbol{E}^-(L)$, contradicting the assumption $\Gamma \vDash \Delta, A_i$.

Valuations α of the free variables do not affect the argument, so covering by L below is to be taken relatively to a given α, which we do not mention, except for $(\forall_R)$ and $(\forall_R^\phi)$.

3. $(\wedge_L)$'. Assume $\Gamma, A_1, A_2 \vDash \Delta$, let semikernel L cover the rule's conclusion, $\Gamma \subseteq L$ and $\Delta \subseteq \boldsymbol{E}^-(L)$. If $A_1 \wedge A_2 \in L$, then $\boldsymbol{E}(A_1 \wedge A_2) = \{\neg A_1, \neg A_2\} \subseteq \boldsymbol{E}^-(L)$, so $\boldsymbol{E}(\{\neg A_1, \neg A_2\}) = \{A_1, A_2\} \subseteq L$, contradicting $\Gamma, A_1, A_2 \vDash \Delta$. Thus $A_1 \wedge A_2 \in \boldsymbol{E}^-(L)$ and $L \vDash \Gamma, A_1 \wedge A_2 \Rightarrow \Delta$.

4. $(\neg_R)$'. Assume $\Gamma, A \vDash \Delta$, let semikernel L cover the rule's conclusion, and assume $\Gamma \subseteq L$ and $\Delta \subseteq \boldsymbol{E}^-(L)$. If $\neg A \in L$, we are done, while if $\neg A \in \boldsymbol{E}^-(L)$ then $A \in L$, which contradicts the assumption, since now $\Gamma \cup \{A\} \subseteq L$ and $\Delta \subseteq \boldsymbol{E}^-(L)$.

5. $(\neg_L)$'. Assume $\Gamma \vDash \Delta, A$, let L cover the rule's conclusion, $\Gamma \subseteq L$ and $\Delta \subseteq \boldsymbol{E}^-(L)$. If $\neg A \in \boldsymbol{E}^-(L)$, we are done, while if $\neg A \in L$ then $A \in \boldsymbol{E}(\neg A) \subseteq \boldsymbol{E}^-(L)$, contradicting the assumption, since now $\Gamma \cup \{A\} \subseteq L$ and $(\Delta \cup \{A\}) \subseteq \boldsymbol{E}^-(L)$.

6. $(\forall_L)$'. Assume $F(t), \Gamma, \forall x F(x) \vDash \Delta$ and let L cover the rule's conclusion. If $\forall x F(x) \notin L$, i.e., $\forall x F(x) \in \boldsymbol{E}^-(L)$, then $L \vDash \Gamma, \forall x F(x) \Rightarrow \Delta$. If $\forall x F(x) \in L$ then also $F(t) \in L$, since $\neg F(t) \in \boldsymbol{E}(\forall x F(x)) \subseteq \boldsymbol{E}^-(L)$ and $\boldsymbol{E}(\neg F(t)) = \{F(t)\}$. As L covers the premise, either $\Gamma \cap \boldsymbol{E}^-(L) \neq \varnothing$, since $F(t) \notin \boldsymbol{E}^-(L)$, or $\Delta \cap L \neq \varnothing$. Either case yields the claim for L, which was arbitrary, so $\Gamma, \forall x F(x) \vDash \Delta$.

7. $(\forall_R)$'. Assume (*) $\Gamma \vDash \Delta, F(y)$, with fresh $y \in o\mathcal{V}$, i.e., $\Gamma \vDash \Delta, F(m)$ for every $m \in M$. Let α be an assignment to $\mathcal{V}(\Gamma, \Delta, \forall x F(x)) \not\ni y$, such that $\alpha(\Gamma) \subseteq L$ and $\alpha(\Delta) \subseteq \boldsymbol{E}^-(L)$. If $\alpha(\forall x F(x)) \notin L$ then $\alpha(\forall x F(x)) \in \boldsymbol{E}^-(L)$ and some $\alpha(\neg F(m)) \in L$, since $\boldsymbol{E}(\alpha(\forall x F(x))) = \{\alpha(\neg F(m)) \mid m \in M\}$. Extending α with $\alpha(y) = m$ yields a contradiction to (*).

8. $(\forall_L^\phi)$'. The argument repeats that for $(\forall_L)$'. Let $\Gamma, F(S), \forall \phi F(\phi) \vDash \Delta$ and L cover the rule's conclusion (under a fixed α). If $\forall \phi F(\phi) \notin L$ then $\forall \phi F(\phi) \in \boldsymbol{E}^-(L)$, yielding $L \vDash \Gamma, \forall \phi F(\phi) \Rightarrow \Delta$. If $\forall \phi F(\phi) \in L$ then also $F(S') \in L$, for each sentence $S' \in \boldsymbol{S}_M^+$, since $\neg F(S') \in \boldsymbol{E}(\forall \phi F(\phi)) \subseteq \boldsymbol{E}^-(L)$ and $\boldsymbol{E}(\neg F(S')) = \{F(S')\}$. Thus L covers also the premise, hence, either $\Gamma \cap \boldsymbol{E}^-(L) \neq \varnothing$, since $F(S) \notin \boldsymbol{E}^-(L)$, or $\Delta \cap L \neq \varnothing$. Either case yields the claim for L, which was arbitrary (as was α), so $\Gamma, \forall \phi F(\phi) \vDash \Delta$.

9. $(\forall_R^\phi)$'. Assume $\Gamma \vDash \Delta, F(\theta)$ for a fresh $\theta \in s\mathcal{V}$, that is, $\Gamma \vDash \Delta, F(S)$ for every $S \in \boldsymbol{S}_M^+$. Let semikernel L cover the rule's conclusion. If $\forall \phi F(\phi) \in L$ then L satisfies it. If $\forall \phi F(\phi) \notin L$ then $\forall \phi F(\phi) \in \boldsymbol{E}^-(L)$, so some $\neg F(S) \in L$, since $\boldsymbol{E}(\forall \phi F(\phi)) = \{\neg F(S) \mid S \in \boldsymbol{S}_M^+\}$. Now L covers also

$\Gamma \Rightarrow \Delta, F(S)$ and $F(S) \notin L$. Since $\Gamma \vDash \Delta, F(S)$, either $\Gamma \cap \boldsymbol{E}^{-}(L) \neq \varnothing$ or $\Delta \cap L \neq \varnothing$. In each case L satisfies the conclusion.

10. S-equality rules are sound (and invertible), keeping unchanged the unifiers of equations between the premise and the conclusion.

(i) For each formula A, each instance $A' \doteq A'$ is satisfied in each (relevant) semikernel (Definition 2.12), hence satisfaction of the premise and of the conclusion of (ref) are equivalent.

(ii) For (rpR)' and (rpL)', let α valuate free variables so that $\alpha(A) \doteq \alpha(B)$, and $\overline{F}$ be the result of instantiating by α the free variables of F other than ϕ. Since $F[\phi\backslash B]$ and $F[\phi\backslash A]$ are legal, all free variables of A/B remain free after these substitutions. Hence: $\alpha(F[\phi\backslash B]) = \overline{F}[\phi\backslash\alpha(B)] = \overline{F}[\phi\backslash\alpha(A)] = \alpha(F[\phi\backslash A])$.

This covers also the case of (rpL)' substituting A for B into equation $L(B) \doteq R(B)$ on the left. If $\alpha(A) \doteq \alpha(B)$ then $L(\alpha(B)) \doteq R(\alpha(B))$ iff $L(\alpha(A)) \doteq R(\alpha(A))$.

(iii) (unif)' is sound because $A_i \doteq B_i$ and $O(A_i) \doteq O(B_i)$ have the same unifiers for every $O(...)$. For every α, $\alpha(A_i) \doteq \alpha(B_i) \Leftrightarrow O(\alpha(A_i)) \doteq O(\alpha(B_i)) \Leftrightarrow \alpha(O(A_i)) \doteq \alpha(O(B_i))$.

Likewise, $\forall vA \doteq \forall vB$ and $A[v\backslash\dot{v}] \doteq B[v\backslash\dot{v}]$ have the same unifiers, so $\alpha(\forall vA) \doteq \alpha(\forall vB) \Leftrightarrow \alpha(A[v\backslash\dot{v}]) \doteq \alpha(B[v\backslash\dot{v}])$, giving soundness (and invertibility) of (unif$^{\forall}$)'. □

The proof of completeness below refers to the following simple consequence of Definition 2.1.

Fact 5.2. *In any language graph G_M, the following relations hold between the form of a non-atomic sentence $X \in \boldsymbol{S}_M$ and forms of its out- and in-neighbours:*

(a) $\boldsymbol{E}^{-}(X) = \{\neg X\}$ *– when X does not start with $\neg$,*
(b) $\boldsymbol{E}^{-}(\neg X) = \{X \wedge S \mid S \in \boldsymbol{S}_M\} \cup \{\neg\neg X, (\forall\phi F(\phi)$ *if* $X \doteq F[\phi\backslash S]), (\forall x F(x)$ *if* $X \doteq F[x\backslash t])\}$
(c) when X does not start with $\neg$, then each out-neighbour of X does,
(d) $\boldsymbol{E}(\neg X) = \{X\}$.
For atomic X, $\boldsymbol{E}^{-}(X) = \{\neg X\} = \boldsymbol{E}(X)$ and $\boldsymbol{E}^{-}(\neg X) = \{X\} = \boldsymbol{E}(\neg X)$.

The proof of completeness applies the standard technique because $\mathsf{LSO}^{\doteq}$ is essentially a first-order reasoning system. A few adjustments are needed for handling the deviations from FOL. We must ensure not only that all formulas are processed and all terms are substituted by $(\forall_L)$', but also that all sentences are substituted by $(\forall_L^{\phi})$'.

Fact 5.3 (2.17). *If $\Gamma \nvdash \Delta$ in $\mathsf{LSO}^{\doteq}$ then there is an $\mathcal{L}^+ \supseteq \mathcal{L}$ and a $G \in \mathcal{LG}r(\mathcal{L}^+)$ with $L \in SK(G)$ such that (i) $\Gamma \subseteq L$ and (ii) $\Delta \subseteq \boldsymbol{E}_G^{-}(L)$, hence L covers $\Gamma \cup \Delta$.*

Proof. Let $\boldsymbol{A}_{\mathcal{V}}$ be all atoms of $\mathcal{L}$ with the subset $Eq \subseteq \boldsymbol{A}_{\mathcal{V}}$ of equations. $\dot{E}q$ are equations using also s-constants $s\dot{\mathcal{V}}$ and terms $\dot{\boldsymbol{T}}_{o\mathcal{V}}$ involving also constants $o\dot{\mathcal{V}}$. $\boldsymbol{F}_{\mathcal{V}}$ are all formulas over atoms $\boldsymbol{A}_{\mathcal{V}}$ and $\dot{\boldsymbol{F}}_{\mathcal{V}} = \boldsymbol{F}_{\mathcal{V}} \cup \dot{E}q$. $\mathcal{L}^+$ includes, as additional s-constants, free $s\mathcal{V}' \subseteq s\mathcal{V}$ from the derivation.

For infinite $\mathcal{L}$, we ensure enough (eigen)variables. For the first regular cardinal $\kappa \geq \kappa' = |\dot{\boldsymbol{F}}_{\mathcal{V}}| \geq \omega$, add enough items of each following kind so that $\kappa = |o\mathcal{V}| = |s\mathcal{V}| = |o\dot{\mathcal{V}}| = |s\dot{\mathcal{V}}|$, hence also $|\dot{\boldsymbol{F}}_{\mathcal{V}}| = \kappa$.

Function Oc lists for equations and non-atomic formulas the forms of their possible occurrences, in the prospective derivation, on the $(\boldsymbol{l})$eft or $(\boldsymbol{r})$ight of $\Rightarrow$, adding for the quantified formulas $(\boldsymbol{l})$ their possible instantiations:

– $Oc(\neg A) = \{(\neg A, \boldsymbol{l}), (\neg A, \boldsymbol{r})\}$, and $Oc(A \wedge B) = \{(A \wedge B, \boldsymbol{l}), (A \wedge B, \boldsymbol{r})\}$,
– $Oc(A \doteq B) = \{(A \doteq B, \boldsymbol{l})\}$ for each equation in $Eq \cup \dot{E}q$,
– $Oc(\forall\phi F\phi) = \{(\forall\phi F\phi, \boldsymbol{l}, A) \in \boldsymbol{F}_{\mathcal{V}} \times \{\boldsymbol{l}\} \times \boldsymbol{F}_{\mathcal{V}} \mid (F\phi)[\phi\backslash A] \text{ legal}\} \cup \{(\forall\phi F\phi, \boldsymbol{r})\}$
– $Oc(\forall x Fx) = \{(\forall x Fx, \boldsymbol{l}, t) \in \boldsymbol{F}_{\mathcal{V}} \times \{\boldsymbol{l}\} \times \boldsymbol{T}_{o\mathcal{V}} \mid (Fx)[x\backslash t] \text{ legal}\} \cup \{(\forall x Fx, \boldsymbol{r})\}$.

Let $Oc(\dot{\boldsymbol{F}}_{\mathcal{V}}) = \bigcup_{A \in \dot{\boldsymbol{F}}_{\mathcal{V}}} Oc(A)$ and, assuming AC, $\boldsymbol{W}$ be a well-ordering of $Oc(\dot{\boldsymbol{F}}_{\mathcal{V}}) \times \kappa$, with each element of $Oc(\dot{\boldsymbol{F}}_{\mathcal{V}})$ occurring cofinally often. $\boldsymbol{W}_Y$ is a well-ordering of $Y \in \{s\mathcal{V}, o\mathcal{V}, s\dot{\mathcal{V}}, o\dot{\mathcal{V}}\}$.

1. In the usual way, we construct a derivation tree starting with the given target sequent $\Gamma \Rightarrow \Delta$ as the root and proceeding along $\boldsymbol{W}$. An *active* sequent – initially, only the root – is a non-axiomatic leaf of the tree constructed bottom-up so far. Each branch of the derivation keeps its own track of the items used from each of $\boldsymbol{W}_{s\mathcal{V}}$, $\boldsymbol{W}_{o\mathcal{V}}$, $\boldsymbol{W}_{s\dot{\mathcal{V}}}$ and $\boldsymbol{W}_{o\dot{\mathcal{V}}}$. At each successor step $n+1$, the action depends on the form of $\boldsymbol{W}_{n+1}$.

1.i. When $\boldsymbol{W}_{n+1} = (A \doteq B, \boldsymbol{l})$, we apply rules for $\doteq$.

(a) If A and B are syntactically identical, we add $A \doteq A$ to the left side of each active sequent.

For $A \doteq B$, with syntactically distinct A, B, occurring on the left of an active sequent, three cases correspond to rules (rpL)', (unif)' and (rpR)':

(b) If also $F[\phi\backslash A]$ (or $F[\phi\backslash B]$) occurs on the left, and both substitutions are legal, we add $F[\phi\backslash B]$ (or $F[\phi\backslash A]$) on the left.

(c) If $A \doteq B$ has the form $O(A') \doteq O(B')$, for any operator/connective O with syntactically distinct A', B', we add $A' \doteq B'$ on the left. If $O = \forall v$, $v \in o\mathcal{V}/s\mathcal{V}$, we add $A'[v\backslash\dot{v}] \doteq B'[v\backslash\dot{v}]$ with the least $\dot{v} \in \boldsymbol{W}_{o\dot{\mathcal{V}}}/\boldsymbol{W}_{s\dot{\mathcal{V}}}$ unused on this branch.

(d) If $F[\phi\backslash A]$ (or $F[\phi\backslash B]$) occurs on the right of an active sequent, with A (or B) not under $\doteq$, then we add there $F[\phi\backslash B]$ (or $F[\phi\backslash A]$), if both substitutions are legal.

Cases in **1.i** ensure that all equational rules are applied in all possible ways in the limit. The remaining cases address non-atomic formulas at the first position in any successor step $\boldsymbol{W}_{n+1}$.

1.ii. For $\boldsymbol{W}_{n+1} = (A \wedge B, \boldsymbol{l})$, every active sequent of the form $A \wedge B, \Gamma \Rightarrow \Delta$ is replaced by

$$\frac{A, B, A \wedge B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} .$$

For $\boldsymbol{W}_{n+1} = (A \wedge B, \boldsymbol{r})$, every active sequent of the form $\Gamma \Rightarrow A \wedge B, \Delta$ is replaced by

$$\frac{\Gamma \Rightarrow A, A \wedge B, \Delta \qquad \Gamma \Rightarrow B, A \wedge B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} .$$

1.iii. For $\boldsymbol{W}_{n+1} = (\neg A, \boldsymbol{l})$ and $\neg A$ occurring on the left of an active sequent, we add $\dfrac{\Gamma, \neg A \Rightarrow \Delta, A}{\Gamma, \neg A \Rightarrow \Delta}$.

For $\boldsymbol{W}_{n+1} = (\neg A, \boldsymbol{r})$ and $\neg A$ occurring on the right of an active sequent, we add $\dfrac{A, \Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg A}$.

1.iv. For $\boldsymbol{W}_{n+1} = (\forall x F x, \boldsymbol{l}, t)$, if all free variables of t occur in an active sequent of the form $\Gamma, \forall x F x \Rightarrow \Delta$, then add a new sequent with $F(t)$ added in the antecedent:

$$\frac{F(t), \Gamma, \forall x F(x) \Rightarrow \Delta}{\Gamma, \forall x F(x) \Rightarrow \Delta} .$$

Active sequents not containing some free variables of t are left unchanged.

1.v. For $\boldsymbol{W}_{n+1} = (\forall x F x, \boldsymbol{r})$, replace each active sequent of the form $\Gamma \Rightarrow \Delta, \forall x F x$ by

$\dfrac{\Gamma \Rightarrow \Delta, \forall x F x, F y}{\Gamma \Rightarrow \Delta, \forall x F x}$ where y is the least element of $\boldsymbol{W}_{o\mathcal{V}}$ unused on this branch.

1.vi. For $\boldsymbol{W}_{n+1} = (\forall \phi F \phi, \boldsymbol{l}, S)$, if all free variables of S occur in an active sequent of the form $\Gamma, \forall \phi F \phi \Rightarrow \Delta$, then replace every such active sequent by

$$\frac{F(S), \Gamma, \forall \phi F \phi \Rightarrow \Delta}{\Gamma, \forall \phi F \phi \Rightarrow \Delta}$$

Active sequents not containing some free variables of S are left unchanged.

1.vii. For $\boldsymbol{W}_{n+1} = (\forall \phi F \phi, \boldsymbol{r})$, each active sequent of the form $\Gamma \Rightarrow \Delta, \forall \phi F(\phi)$ is replaced by

$\dfrac{\Gamma \Rightarrow \Delta, \forall \phi F \phi, F \alpha}{\Gamma \Rightarrow \Delta, \forall \phi F \phi}$ α is the least element of $\boldsymbol{W}_{s\mathcal{V}}$ unused on this branch.

2. A branch gets closed when its leaf is an axiom, and the tree closes when all branches do, yielding a proof. Otherwise, either some branch terminates with a non-axiomatic irreducible sequent, or the tree is obtained as the ω-limit of this process. Since $\boldsymbol{F}_{\mathcal{V}}$ can be uncountable, the saturation process may require transfinite iterations. Each branch remaining open at $\boldsymbol{W}_\lambda$, for any limit ordinal λ, gives rise to active sequent $\Gamma_\lambda \Rightarrow \Delta_\lambda$, with $\Gamma_\lambda/\Delta_\lambda$ gathering the left/right sides of the sequents on the branch ($\Gamma_\lambda = \bigcup_{\iota<\lambda} \Gamma_\iota$, and $\Delta_\lambda = \bigcup_{\iota<\lambda} \Delta_\iota$). The process continues then with $\boldsymbol{W}_{\lambda+1}$, until either all branches get closed or $\boldsymbol{W}$ becomes exhausted.

When the process terminates, on each open branch, order-isomorphic to $\boldsymbol{W}$, every occurring formula is processed by the appropriate rule. Each variable $v \in \mathcal{V}$ is introduced by the right rule at most once – $o\mathcal{V}', s\mathcal{V}'$ are the subsets of $o\mathcal{V}, s\mathcal{V}$ actually occurring unbound on β. All instances of the quantified formulas, also with $x \in o\mathcal{V}', \theta \in s\mathcal{V}'$, are introduced on the left side.

Any finite non-axiomatic branch gives easily a countermodel. We show that, upon completion, also an infinite (possibly transfinite) open branch β gives a countermodel to all sequents on this branch. As β has no (Ax), we often use implicitly $\beta_L \cap \beta_R = \varnothing$, and as it has no (uniAx), each subset of equations on the left is unifiable. First, consider only equations in β.

3. Let β_L/β_R be all formulas – and Eq_L/Eq_R equations – occurring in β on the left/right of $\Rightarrow$. We argue that Eq_L contains a unifier U of Eq_L that does not unify any equation from Eq_R.

Each trivial $A \doteq A$ is in Eq_L by **1.i.**(a), so no such occurs on the right – in each $(S \doteq T) \in Eq_R$, S and T are syntactically distinct. Each finite subset of Eq_L is unifiable since the branch has no (uniAx). Consequently, Eq_L is unifiable, since unifiability has finite character: a failure of the systematic unification process **1.i.**(b)-(c), reflecting applications of (rpL)' and (unif)'-(unif$^\forall$)', happens in a finite time (cf. [1, Property 4]). In fact, Eq_L contains its unifier. If $(O(A) \doteq O(B)) \in Eq_L$ then also $A \doteq B$ by **1.i.**(c), and likewise for $(\forall v A v \doteq \forall v B v) \in Eq_L$. This continues to the trivial equations $C \doteq C$, or $\phi \doteq C$ with $\phi \in \mathcal{V}$, and the latter give a unifier $U \subseteq Eq_L$. (Any other occurrence of ϕ in $(A(\phi) \doteq B(\phi)) \in Eq_L$ has the accompanying instance $(A(C) \doteq B(C)) \in Eq_L$ by **1.i.**(b).) Any $\dot{v} \in \dot{\mathcal{V}}$ occurring in β occurs only in equations on the left, which are unifiable by U with the trivial $\dot{v} \doteq \dot{v}$. Hence, truth-values of $\dot{\mathcal{V}}$ do not matter and they can be interpreted, for instance, identically to their image under any functions $s\dot{\mathcal{V}} \to s\mathcal{V}'$ and $o\dot{\mathcal{V}} \to o\mathcal{V}'$.

If $(S \doteq T) \in Eq_R$ then S and T are not unifiable by equations from Eq_L. For suppose that they are, e.g., $S \doteq T$ is $P(\phi) \doteq P(Q)$, while equation $\phi \doteq Q$ enters Eq_L at some stage. $P(Q) \doteq P(Q)$ enters Eq_L by **1.i.**(a), and then also $P(\phi) \doteq P(Q)$ by **1.i.**(b), giving (Ax). (If $(S \doteq T) \in Eq_R$ is unifiable but not by equations from Eq_L then S and T are syntactically distinct. All $\mathcal{V}$ contained in S/T become closed in the countermodel, point **4**, making $S \not\doteq T$.)

For each $(\theta \doteq S) \in U$ with $\theta \in s\mathcal{V}'$, any other formula $F(\theta) \in \beta_L$ with free θ has instance $F(S) \in \beta_L$ by **1.i.**(b), when $F[\theta\backslash S]$ is legal. By **1.i.**(d), the same holds for the occurrences of θ on the right, that are not under $\doteq$. Thus, applying legally substitutions from U to all formulas (including also equations on the right and $\theta \doteq S$ on the left), leaves only trivial equations on the left and only false ones on the right. No new formulas are added to β except, possibly, false instances $F(S)$ of equations $F(\theta) \in \beta_R$, when $(\theta \doteq S) \in U$.

4. Free $s\mathcal{V}'$, possibly remaining on β, are added to the final language $\mathcal{L}^+ = \mathcal{L} \cup s\mathcal{V}'$ as s-constants and the argument continues with β resulting from the substitutions in **3**. If it contains no FOL-atoms, we let $M = \varnothing$. Otherwise, we first construct in the standard way a FOL-structure M that gives a countermodel to $(\beta_L \cap \boldsymbol{S}_M^-) \Rightarrow (\beta_R \cap \boldsymbol{S}_M^-)$. For $G = G_M(\mathcal{L}^+)$ we show that

(def) $L = \beta_L \cup Z$, where $Z = (\boldsymbol{E}(\beta_R) \cap \boldsymbol{E}^-(\beta_R)) \setminus \beta_L$,

is a required semikernel of G, namely, such that $\beta_L \subseteq L$ and $\beta_R \subseteq \boldsymbol{E}^-(L)$.

By point **3**, β_L contains only trivial equations, while β_R only $S \doteq T$ for S, T that are syntactically distinct under α, so L satisfies the requirement for the intended interpretation of $\doteq$.

By (def) $\beta_L \subseteq L$ and $\boldsymbol{E}(\beta_R) \cap \boldsymbol{E}^-(\beta_R) \subseteq L$, but we have to verify $\beta_R \subseteq \boldsymbol{E}^-(L)$. Indeed, there is no $X \in \beta_R \setminus \boldsymbol{E}^-(L) = \beta_R \setminus \big(\boldsymbol{E}^-(\boldsymbol{E}^-(\beta_L)) \cup \boldsymbol{E}^-(\boldsymbol{E}(\beta_R) \cap \boldsymbol{E}^-(\beta_R))\big)$, for if $X \in \beta_R$ is:

1. an atom, then $\boldsymbol{E}(X) \cap \boldsymbol{E}^-(X) = \{\neg X\} \subseteq Z \subseteq L$;
2. $\neg Y$, then $Y \in \beta_L$ and $X \in \boldsymbol{E}^-(L)$;
3. $A_1 \wedge A_2$, then some $A_i \in \beta_R$ and $\neg A_i \in \boldsymbol{E}(X) \cap \boldsymbol{E}^-(A_i) \subseteq Z \cup \beta_L$, so $X \in \boldsymbol{E}^-(L)$;
4. $\forall \phi F \phi$, then $F\psi \in \beta_R$, for some $\psi \in s\mathcal{V}$, and $\neg F\psi \in \boldsymbol{E}(X) \cap \boldsymbol{E}^-(F\psi) \subseteq Z \cup \beta_L$.
5. $\forall y F y$, then $Fx \in \beta_R$, for some $x \in o\mathcal{V}$, and $\neg Fx \in \boldsymbol{E}(X) \cap \boldsymbol{E}^-(Fx) \subseteq Z \cup \beta_L$.

5. We show $\boldsymbol{E}(L) \subseteq \boldsymbol{E}^-(L) \cap (\boldsymbol{V} \setminus L)$ separately for each part β_L and Z of L. The subpoints below establish $\boldsymbol{E}(\beta_L) \subseteq \boldsymbol{E}^-(L) \cap (\boldsymbol{V} \setminus L)$, considering cases of $A \in \beta_L$.

5.i. For an atom $A \in \boldsymbol{A}_M$, since $A \in \beta_L \subseteq L$ so $A \notin \beta_R$, hence $\neg A \notin \beta_L$ and, since $\boldsymbol{E}(\neg A) \overset{5.2}{=} \{A\}$, $\neg A \notin \boldsymbol{E}^-(\beta_R)$. Thus $\boldsymbol{E}(A) \overset{5.2}{=} \{\neg A\} \subseteq \boldsymbol{E}^-(A) \cap (\boldsymbol{V} \setminus L) \subseteq \boldsymbol{E}^-(L) \cap (\boldsymbol{V} \setminus L)$.

5.ii. $A = \neg C \in \beta_L$ implies $C \in \beta_R$, so $\boldsymbol{E}(A) \overset{5.2}{=} \{C\} \subseteq \beta_R \subseteq \boldsymbol{E}^-(L)$ by **4**.

We show $\boldsymbol{E}(A) \subset \boldsymbol{V} \setminus L$. $C \notin \beta_L$ since $\beta_L \cap \beta_R = \varnothing$. Suppose $C \in \boldsymbol{E}(\beta_R) \cap \boldsymbol{E}^-(\beta_R)$. If $C = \neg D$ then $\neg D \in \boldsymbol{E}^-(\beta_R)$, i.e., $\boldsymbol{E}(\neg D) \overset{5.2}{=} \{D\} \subset \beta_R$, while $A = \neg C = \neg\neg D \in \beta_L$ implies also $\neg D \in \beta_R$ and $D \in \beta_L$, contradicting $\beta_L \cap \beta_R = \varnothing$.

Otherwise, i.e., if C does not start with $\neg$, then for any $F \in \beta_R$ for which $C \in \boldsymbol{E}(F)$, Fact 5.2.((c)-(d)) forces $F = \neg C = A$, contradicting $\beta_R \cap \beta_L = \varnothing$.

5.iii. $A = B \wedge C \in \beta_L$ implies $\{B, C\} \subset \beta_L$ and $\{\neg B, \neg C\} \cap \beta_L = \varnothing$, so $\boldsymbol{E}(B \wedge C) \overset{5.2}{=} \{\neg B, \neg C\} \subseteq \boldsymbol{V} \setminus \beta_L$ and $\boldsymbol{E}(B \wedge C) = \{\neg B, \neg C\} \overset{5.2}{\subseteq} \boldsymbol{E}^-(\{B, C\}) \subseteq \boldsymbol{E}^-(\beta_L)$. If, say, $\neg B \in \boldsymbol{E}^-(\beta_R)$, then $B \in \beta_R$ would contradict $\beta_L \cap \beta_R = \varnothing$. The same if $\neg C \in \boldsymbol{E}^-(\beta_R)$. Thus, $\boldsymbol{E}(B \wedge C) \subseteq \boldsymbol{E}^-(L) \cap (\boldsymbol{V} \setminus L)$.

5.iv. $A = \forall \phi F(\phi) \in \beta_L \Rightarrow \{F(S) \mid S \in \boldsymbol{S}_M\} \subseteq \beta_L$, so $\boldsymbol{E}(\forall \phi F(\phi)) \overset{5.2}{=} \{\neg F(S) \mid S \in \boldsymbol{S}_M\} \overset{5.2}{\subseteq} \boldsymbol{E}^-(\{F(S) \mid S \in \boldsymbol{S}_M\}) \subseteq \boldsymbol{E}^-(\beta_L) \subseteq \boldsymbol{E}^-(L)$.

If any $\neg F(S) \in L$ then either $\neg F(S) \in \beta_L$, so $F(S) \in \beta_R$, or $\neg F(S) \in \boldsymbol{E}(\beta_R) \cap \boldsymbol{E}^-(\beta_R)$, which implies $F(S) \in \beta_R$, since $\boldsymbol{E}(\neg F(S)) \overset{5.2}{=} \{F(S)\}$. In either case, $F(S) \in \beta_R$ contradicts $\beta_L \cap \beta_R = \varnothing$. Thus $\boldsymbol{E}(\forall \phi F(\phi)) \subseteq \boldsymbol{V} \setminus L$.

5.v. For $A = \forall x F(x)$, the argument is as in **5.iv**. $\forall x F(x) \in \beta_L$ implies $\{F(t) \mid t \in \boldsymbol{T}_M\} \subseteq \beta_L$, so $\boldsymbol{E}(\forall x F(x)) \overset{5.2}{=} \{\neg F(t) \mid t \in \boldsymbol{T}_M\} \overset{5.2}{\subseteq} \boldsymbol{E}^-(\{F(t) \mid t \in \boldsymbol{T}_M\}) \subseteq \boldsymbol{E}^-(\beta_L) \subseteq \boldsymbol{E}^-(L)$.

If any $\neg F(t) \in L$, then either $\neg F(t) \in \beta_L$, so $F(t) \in \beta_R$, or $\neg F(t) \in \boldsymbol{E}(\beta_R) \cap \boldsymbol{E}^-(\beta_R)$, which implies $F(t) \in \beta_R$, since $\boldsymbol{E}(\neg F(t)) \overset{5.2}{=} \{F(t)\}$. In either case, $F(t) \in \beta_R$ contradicts $\beta_L \cap \beta_R = \varnothing$. Thus $\boldsymbol{E}(\forall x F(x)) \subseteq \boldsymbol{V} \setminus L$.

6. Also for each sentence $S \in Z = (\boldsymbol{E}(\beta_R) \cap \boldsymbol{E}^-(\beta_R)) \setminus \beta_L$ it holds that $\boldsymbol{E}(S) \subseteq \boldsymbol{E}^-(L) \cap (\boldsymbol{V} \setminus L)$:

6.i. If $S \in Z$ does not start with $\neg$, then $\boldsymbol{E}^-(S) \overset{5.2}{=} \{\neg S\}$, so $\neg S \in \beta_R$, implying $S \in \beta_L$, so $S \notin Z$.

6.ii. If $S = \neg A \in Z \subseteq \boldsymbol{E}^-(\beta_R)$ then $\boldsymbol{E}(\neg A) \overset{5.2}{=} \{A\} \subset \beta_R \overset{\mathbf{4}}{\subseteq} \boldsymbol{E}^-(L)$. If $A \in Z$, then by **6.i** it starts with $\neg$, i.e., $A = \neg B$ and $\boldsymbol{E}(\neg B) \overset{5.2}{=} \{B\} \subset \beta_R$. Since also $A \in \beta_R$ so $B \in \beta_L$, contradicting $\beta_L \cap \beta_R = \varnothing$. Hence $A \notin Z$ and $A \notin \beta_L$ (since $A \in \beta_R$), i.e., $A \notin L = Z \cup \beta_L$, so that $\boldsymbol{E}(\neg A) = \{A\} \subseteq \boldsymbol{V} \setminus L$.

By points **5** and **6**, $\boldsymbol{E}(L) = \boldsymbol{E}(\beta_L) \cup \boldsymbol{E}(Z) \subseteq \boldsymbol{E}^-(L) \cap (\boldsymbol{V} \setminus L)$, so $L \in SK(G)$. By **3** it contains trivial equations and negations of the false ones, thus respecting the intended interpretation of $\doteq$, and by **4** satisfies (i) $\Gamma \subseteq L$ and (ii) $\Delta \subseteq \boldsymbol{E}^-_G(L)$. □

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REFERENCES

[1] Wan Fokkink, *Unification for infinite sets of equations between finite terms*, ***Information Processing Letters***, vol. 62 (1997), pp. 183–188.

[2] Hannes Leitgeb, *What truth depends on*, ***Journal of Philosophical Logic***, vol. 34 (2005), pp. 155–192.

[3] Victor Neumann-Lara, *Seminúcleos de una digráfica*, ***Technical report***, Anales del Instituto de Matemáticas II, Universidad Nacional Autónoma México, 1971.

[4] Douglas Patterson, *Inconsistency theories of semantic paradox*, ***Philosophy and Phenomenological Research***, vol. 79 (2009), no. 2, pp. 387–422.

[5] Alfred Tarski, *The concept of truth in formalised languages*, ***Logic, semantics, metamathematics, papers from 1923 to 1938***, Oxford at the Clarendon Press, 1956, [translation of the Polish original from 1933 by J. H. Woodger].

[6] Willard van Orman Quine, *Two dogmas of empiricism*, ***The Philosophical Review***, vol. 60 (1951), pp. 20–43.

[7] Michal Walicki, *Paradoxes versus contradictions in a logic of sentential operators*, ***Logic and Logical Philosophy***, (2024), pp. 1–44.

[8] ———, *The structure of paradoxes in a logic of sentential operators*, ***Journal of Philosophical Logic***, vol. 53 (2024), pp. 1579–1639.

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