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RESOLVING INFINITARY PARADOXES

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Abstract. Graph normal form, GNF, [1], was used in [2, 3] for analysing paradoxes in propositional discourses, with the semantics – equivalent to the classical one – defined by kernels of digraphs. The paper presents infinitary, resolution-based reasoning with GNF theories, which is refutationally complete for the classical semantics. Used for direct (not refutational) deduction it is not explosive and allows to identify in an inconsistent discourse, a maximal consistent subdiscourse with its classical consequences. Semikernels, generalizing kernels, provide the semantic interpretation.

§1. Motivation and overview. An informal discourse, represented by just writing its statements in some logical language, can be analyzed for consistency or validity, but hardly for paradoxicality. For paradox does not amount to the inconsistency of the discourse but of its truth-theory, which means here, roughly, the collection of T-schemata for discourse's statements, [3]. There is nothing paradoxical about $a \wedge \neg a$. Its propositional T-schema, $f \leftrightarrow (a \wedge \neg a)$, is unproblematic, classifying this statement, called now f, as false. When there are no references between statements, the truth-theory becomes such a trivially satisfiable repetition of each statment in an equivalence to its unique identifier. When statements refer to statements, identifiers become essential already for their representation. The truth-teller becomes at once $t \leftrightarrow t$, the liar $l \leftrightarrow \neg l$, and the truth-theory may become inconsistent.

Classical provability of everything from such an inconsistent theory makes all its statements, so to speak, equally paradoxical. This is easily found unsatisfactory. The discourse D, to the left below, consists of Yablo's paradox and three statements (a)-(c). Its truth-theory T is given to the right:

(Y)	Yablo's paradox	$\{y_i \leftrightarrow \bigwedge_{j>i} \neg y_j \mid i \in \mathbb{N}\}$
(a)	All statements in (Y) are false.	$a \leftrightarrow \bigwedge_{i\in\mathbb{N}} \neg y_i$
(b)	All statements in (Y) and (c) are false.	$b \leftrightarrow (\neg c \wedge \bigwedge_{i\in\mathbb{N}} \neg y_i)$
(c)	Earth is round.	$c \leftrightarrow \mathbf{1}$ ((c) is true)

One can accept that (a) is a paradox because of (Y), though even this could be disputed. It is a bit harder to accept paradoxicality of (b) which, denying a true claim (c), can be considered false, irrespectively of (Y). But even granting that (b) is a (part of the) paradox, too, there seems to be no reason whatsoever why Yablo's paradox should affect also the indisputability of Earth's roundness.

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The reasoning system **RIP**, presented in Section 3, works with clausal representation of propositional theories like T, using a variation of (positive and negative) hyper-resolution. It is sound and refutationally complete for the classical semantics of countable theories in infinitary logic, Section 4. Thus, each discourse, having inconsistent truth-theory T expressible in this language, can be proven paradoxical by deriving from T the empty clause, $T \vdash \{\}$.

A surprising, paraconsistent effect is achieved by proving consequences in **RIP** directly, instead of refutationally: to check if A follows from T, we try to prove $T \vdash A$ and not $T, \neg A \vdash \{\}$, Section 5. Consequently, weakening is no longer admissible and, with it, neither is *Ex Falso Quodlibet*. The system remains complete for nonredundant clauses, i.e., if $T \models C$, then $T \vdash B$ for some $B \subseteq C$.

For our $T : T \vdash \{\}$, $T \vdash c$ and $T \vdash \neg b$, but neither $T \vdash \neg c$ nor $T \vdash b$. Only for atoms invovled into paradox, like all y_i, we have both $T \vdash y_i$ and $T \vdash \neg y_i$. We can then follow spreading of paradox through the discourse along such atoms, whose both literals are provable, Section 5.2. In our T, this happens only to a.

A paradox appears when truth seems to imply falsehood and vice versa. Identification of statements involved into a paradox by the classical provability of both their truth and falsehood, seems therefore quite satsfactory. Importantly, this does not lead to any semantic dialetheism. Paradox is a failure – inconsistency – of discourse's truth-theory. The statements involved into this failure are characterized by the provability of both literals. Knowing the culprits, there is no need for attaching to them any value – they are simply excluded from semantic interpretation. **RIP** classifies a discourse as one of the three types and, in case (3), draws the demarcation line:

1. The discourse is nonparadoxical, its truth-theory is consistent.
2. All statements of the discourse participate in the paradox.
3. Only a part of the discourse is involved into paradox, like (Y), (a) of D.

Following [1, 3], semantics, given in Section 2, uses digraph kernels and coincides with the classical one in cases (1) and (2). In case (3), kernel semantics generalizes to semikernels, which are kernels of subgraphs without the paradoxical part, Section 5.1, and to which the same reasoning applies. Rendering the syntactic theory as a digraph (and the semantics as its (semi)kernels), opened in [2, 3] a fruitful way to investigate patterns of paradoxes, in particular, of circularity. The present paper touches upon this but, primarily, introduces the reasoning system **RIP**.

§2. Background. A propositional formula is in graph normal form, GNF, when it has the form

$$x \leftrightarrow \bigwedge_{i \in I_x} \neg y_i, \tag{2.1}$$

where all x, y_i are atoms (propositional variables). When $I_x = \varnothing$, this is identified with x. A theory is in GNF when all its formulae are in GNF and every atom occurs in such a formula exactly once unnegated, i.e., exactly once on the left of $\leftrightarrow$.[1] A *discourse* is a theory in GNF and *paradox* is defined as an inconsistent discourse. Plausibility of this definition, implicit in [2], was argued and exemplified in [3], so we give only one illustration.

EXAMPLE 2.2. *Let Θ_1 be the following discourse:*

a. This and the next statement are false.	$a \leftrightarrow \neg a \wedge \neg b$
b. The next statement is false.	$b \leftrightarrow \neg c$
c. The previous statement is false.	$c \leftrightarrow \neg b$

Making b true and a and c false, gives a model, so that Θ_1 does not involve any paradox. Adding the fourth statement:

d. This and the previous statement are false. $\quad d \leftrightarrow \neg d \wedge \neg c$

gives the discourse Θ_2, where paradox is unavoidable.

GNF is indeed a normal form, [1]: every theory in (infinitary) propositional logic $\mathcal{L}_\kappa$ has an equisatisfiable one in GNF.[2] Semantics is defined in the standard way and thus, although focusing on the paradoxical character of discourses, we address indirectly the consistency in infinitary logic in general.

The standard semantics has an equivalent formulation in terms of graph kernels, [2, 3], which will enable a seamless transition between the classical and less classical logic. A graph[3] is a pair $\mathsf{G} = \langle G, \mathsf{N} \rangle$, where $\mathsf{N} \subseteq G \times G$ is also viewed as a set-valued function $\mathsf{N}(x) = \{y \in G \mid \mathsf{N}(x,y)\}$. $\mathsf{N}^{\smile}(X) = \{y \in G \mid x \in \mathsf{N}(x)\}$ is the converse relation to N, and all such set-valued functions are extended pointwise to sets, i.e., $\mathsf{N}(X) = \bigcup_{x \in X} \mathsf{N}(x)$, etc. A *kernel* of a graph G is a subset $K \subseteq G$ which is independent (no edges between vertices in K) and dominating (every vertex in $G \setminus K$ has an edge to some vertex in K), namely, such that $\mathsf{N}^{\smile}(K) = G \setminus K$. $Ker(\mathsf{G})$ denotes kernels of G.

Theories and graphs can be transformed into each others, along with the associated models and kernels. A theory Γ in GNF gives rise to a graph $\mathcal{G}(\Gamma)$ with all atoms as vertices and with edges from every x on the left-hand side of a GNF formula in Γ, to each y_i on its righ-hand side, i.e., $\mathsf{N} = \{\langle x, y_i \rangle \mid x \in G, i \in I_x\}$. For instance, the discourse Θ_1 from Example 2.2, has the graph $\mathcal{G}(\Theta_1) : \circlearrowright a \rightarrow b \rightleftarrows c$.

For a graph $\mathsf{G} = \langle G, \mathsf{N} \rangle$, its theory is $\mathcal{T}(\mathsf{G}) = \{x \leftrightarrow \bigwedge_{y \in \mathsf{N}(x)} \neg y \mid x \in G\}$. (When x is a *sink*, $\mathsf{N}(x) = \varnothing$, this becomes $x \leftrightarrow \top$, i.e., x is included in $\mathcal{T}(\mathsf{G})$.) The two are inverses, so we ignore usually the distinction between theories (in GNF) and graphs, viewing them as alternative presentations. Typically, Γ denotes such a theory or a graph, while G the corresponding set of atoms/vertices.

The presentations are equivalent also semantically: for corresponding graph and theory, the kernels of the former and models of the latter are in bijection. Kernel of a graph G can be defined equivalently as a partition α of G into two

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[1] The formula $a \leftrightarrow \neg b$ is in GNF but the theory $\{a \leftrightarrow \neg b\}$ is not, due to the loose b. Such cases can be treated as abbreviations for GNF theories, here, with a fresh atom $\underline{b}$ and two additional formulae $b \leftrightarrow \neg \underline{b}$ and $\underline{b} \leftrightarrow \neg b$.

[2] $\mathcal{L}_\kappa$ denotes propositional language with formulae of finite depth, formed over an arbitrary set of atoms by unary negation and (possibly) infinite conjunctions of sets of formulae with cardinality $< \kappa$. Binary connectives, such as $\leftrightarrow$, are encoded (but could be added).

[3] In this paper, "graph" means always "directed graph".

disjoint subsets $\langle \alpha^{\mathbf{1}}, \alpha^{\mathbf{0}} \rangle$ such that $\forall x \in G$:

$$(2.3) \qquad \begin{array}{lrcl} (a) & x \in \alpha^{\mathbf{1}} & \Leftrightarrow & \forall y \in \mathsf{N}(x) : y \in \alpha^{\mathbf{0}} \\ (b) & x \in \alpha^{\mathbf{0}} & \Leftrightarrow & \exists y \in \mathsf{N}(x) : y \in \alpha^{\mathbf{1}}. \end{array}$$

Conditions (a) and (b) are equivalent for total α (with $\alpha^{\mathbf{0}} = G \setminus \alpha^{\mathbf{1}}$), so one will suffice, until we consider partial structures. A total α satisfies (2.3) iff $\alpha^{\mathbf{1}} \in Ker(\mathsf{G})$. On the other hand, satisfaction of (2.3) at every $x \in G$ is equivalent to the satisfaction of the respective GNF theory $\mathcal{T}(\mathsf{G})$. So, for corresponding graph and theory, we identify also kernels of the former and models of the latter.

Example 2.4. *The graphs for the discourses from Example 2.2 are:*
$\mathcal{G}(\Theta_1) : a \to b \rightleftarrows c \qquad \mathcal{G}(\Theta_2) : a \to b \rightleftarrows c \leftarrow d$
In $\mathcal{G}(\Theta_1)$, the partition $\alpha = \langle \{b\}, \{a, c\} \rangle$ is the only one satisfying (2.3), i.e., $\alpha^{\mathbf{1}} = \{b\}$ determines the only model of Θ_1/kernel of $\mathcal{G}(\Theta_1)$.

In $\mathcal{G}(\Theta_2)$, the same α satisfies (2.3) at $\{a, b, c\}$, but leaves no satisfying assignment at d. Letting, on the other hand, $\beta^{\mathbf{1}} = \{c\}$ and $\beta^{\mathbf{0}} = \{b, d\}$ satisfies (2.3) at $\{b, c, d\}$, but leaves no possible assignment to a. The graph has no kernel, i.e., the discourse is paradoxical.

The inference system presented below is essentially (negative and positive) hyper-resolution, handling infinitary clausal theories arising from GNF. The two implications in (2.1) give two kinds of clauses for every $x \in G$:

OR-clause: $x \vee \bigvee_{i \in I_x} y_i$, written as $xy_1y_2...$

NAND-clauses: $\neg x \vee \neg y_i$, for every $i \in I_x$, written with overbars, $\overline{x_i x_j}$.

In terms of a graph, the theory contains, for every $x \in G$, the OR-clause $\mathsf{N}[x] = \{x\} \cup \mathsf{N}(x)$ and for every $y \in \mathsf{N}(x)$, the NAND-clause $\overline{xy}$. For the graphs from Example 2.4, the resulting clausal theories are:

$\Theta'_1 = \{ab, bc, \overline{ab}, \overline{bc}, \overline{a}\}$ and $\Theta'_2 = \{ab, bc, cd, \overline{ab}, \overline{bc}, \overline{cd}, \overline{a}, \overline{d}\}$

We treat both kinds of clauses as sets of atoms, and overbars mark only that a set is a NAND-clause. We can therefore write, e.g., $\overline{xy} \subseteq \overline{xyzu}$. $A \subseteq G$ denotes (also) an OR-clause, $\overline{A} = \{\overline{a} \mid a \in A\}$ a NAND-clause, while $\ddot{A}$ either. Sets of unary clauses are denoted $A^+ = \{\{a\} \mid a \in A\}$ and $A^- = \{\{\overline{a}\} \mid a \in A\}$. The considered language contains no mixed, but only OR and NAND, clauses.

Semantics is classical but we encounter also partial structures consisting of two disjoint subsets of G, $\langle P, N \rangle$, with satisfaction defined for $A \subseteq G$:

$\langle P, N \rangle \models A$ iff $P \cap A \neq \varnothing$ and $\langle P, N \rangle \models \overline{A}$ iff $N \cap A \neq \varnothing$.

For any $M \subseteq G$, the total structure $\alpha_M = \langle M, G \setminus M \rangle$ is a classical model iff it satisfies all clauses (for a graph, (2.3)). It can be also seen as α_M, where $M \subseteq G$

– is a *transversal* of OR (for every $P \in$ OR : $M \cap P \neq \varnothing$),

– not containing any NAND (for every $N \in$ NAND : $N \not\subseteq M$).

Equivalently, a model can be given as $\alpha_{G \setminus N}$ for a subset $N \subseteq G$ which is a transversal of NAND, not containing any $P \in$ OR. We record this simple fact ($Tr(S)$ denotes the set of all transversals of S.)

Fact 2.5. *For every $\Gamma =$ OR + NAND, the three sets are in bijection:*

1. $Mod(\Gamma) = \{M \subseteq G \mid \langle M, G \setminus M \rangle \models \Gamma\}$
2. $\{Pt \in Tr(\text{OR}) \mid \forall N \in \text{NAND} : N \not\subseteq Pt\}$
3. $\{Nt \in Tr(\text{NAND}) \mid \forall P \in \text{OR} : P \not\subseteq Nt\}$

§3. Infinitary resolution. Of primary interest to us are graphs (GNF theories) but several results hold for theories with finite NAND-clauses. Saying "every Γ", we mean such theories. The following system **RIP** is complete for such theories with countable OR set, denoted C-F, while it is sound for arbitrary theories (also with infinite NANDs, which we do not consider.)

$$(Ax) \quad \Gamma \vdash C, \quad \text{for } C \in \Gamma$$

$$(Rneg) \quad \frac{\{\Gamma \vdash \overline{a_i A_i} \mid i \in I\} \quad \Gamma \vdash \{a_i \mid i \in I\}}{\Gamma \vdash \overline{\bigcup_{i \in I} A_i}}$$

$$(Rpos) \quad \frac{\Gamma \vdash A \quad \{\Gamma \vdash B_i K_i \mid i \in I\} \quad \{\Gamma \vdash \overline{a_i k} \mid i \in I, k \in K_i\}}{\Gamma \vdash (A \setminus \{a_i \mid i \in I\}) \cup \bigcup_{i \in I} B_i}$$

The rule (Rneg) derives a NAND from NANDs, using a single OR as a side formula, while (Rpos) derives an OR from ORs, using NANDs as side formulae. In (Rneg), $\overline{a_i A_i}$ denotes the NAND $\overline{\{a_i\} \cup A_i}$, where A_i may be empty. These negative premises are "joined" – into the union of all $\overline{A}_i$ – by the OR-clause O, with each $a_i \in O$ belonging to one $\overline{a_i A_i}$.

In (Rpos), among the OR-premises there is the "main" clause A, containing a subset $\{a_i \mid i \in I\}$ such that for each a_i, there is an OR-premise $B_i K_i$ ($B_i \cup K_i$), with side premises $\overline{a_i k}$ for all $k \in K_i$. The conclusion joins the OR-clauses removing the atoms from the negative premises. A special case of the rule has only the main OR-premise A with the side premises $\Gamma \vdash \overline{a}_i, i \in I$, yielding the conclusion $A \setminus \{a_i \mid i \in I\}$.

There are no cardinality restrictions on the index sets I, so finitary logic is an obvious special case. Proofs are well-founded trees with (Ax) at the leafs, rule applications at all internal nodes, and the conclusion at the root. In particular, every branch of a proof is finite.

A pair of examples of diagnosing the paradox by proving the empty clause $\{\}$, may be in order. The side premises are written as side conditions.

In Yablo graph $\langle \mathbb{N}, < \rangle$, ORs are $O_i = \{j \mid j \geq i\}$ for all $i \in \mathbb{N}$, and NANDs all pairs $\overline{ij}$, for $i \neq j$. For each i, starting with the axioms $\overline{ij}$ for all $j > i$ and using O_{i+1}, yields $\overline{i}$, and from these $\{\}$ follows using O_1:

$$\frac{\dfrac{\overline{12}, \overline{13}, \overline{14}, \ldots}{\overline{1}}\, O_2 \quad \dfrac{\overline{23}, \overline{24}, \overline{25}, \ldots}{\overline{2}}\, O_3 \quad \ldots \quad \dfrac{\{\overline{ij} \mid j > i\}}{\overline{i}}\, O_{i+1} \quad \ldots}{\{\}}\, O_1$$

A "3-Yablo", Fig. 1, with each edge $i \to j$ from the Yablo graph, for $j > i+1$ (i.e., except those along the "main" ray), stretched to an odd path of, say, length 3: $y_i \to a^i_j \to b^i_j \to y_j$.

D1. for all $i < j : \overline{y_i y_j}$, e.g.: $\dfrac{\overline{y_1 a^1_3} \quad \overline{b^1_3 y_3}}{\overline{y_1 y_3}}\, a^1_3 b^1_3$

D2. for all $i, k \geq i+2 : \overline{y_i a^{i+1}_k}$, e.g.: $\dfrac{\overline{y_1 a^1_4} \qquad \dfrac{\overline{a^2_4 b^2_4} \quad \overline{b^1_4 y_4}}{\overline{a^2_4 b^1_4}}\, b^{2:4} y_4}{\overline{y_1 a^2_4}}\, a^1_4 b^1_4$

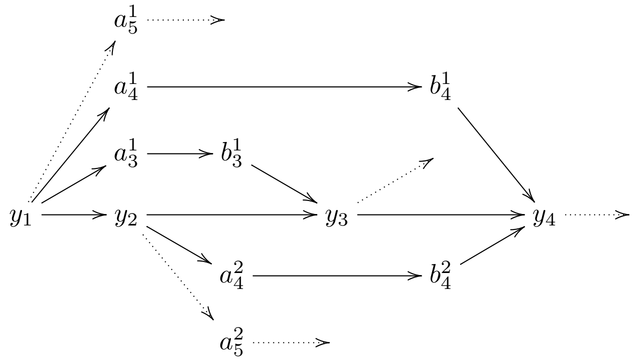

FIGURE 1. "3-Yablo" graph

D3. for all $i : \overline{y}_i$, e.g.: $$\dfrac{\dfrac{}{\overline{y_1y_2}} \quad \dfrac{D1}{\overline{y_1y_3}} \quad \dfrac{D2}{\overline{y_1a_4^2}} \quad \dfrac{D2}{\overline{y_1a_5^2}} \quad \ldots}{\overline{y}_1}\ y_2y_3 \cup \{a_k^2 \mid k \geq 4\}$$

D4. for all $j : \overline{a}_j^1$, e.g.: $$\dfrac{\dfrac{}{\overline{a_3^1b_3^1}} \quad \dfrac{D3}{\overline{y}_3}}{\overline{a}_3^1}\ b_3^1y_3$$

D5. $$\dfrac{\dfrac{D3}{\overline{y}_1} \quad \dfrac{D3}{\overline{y}_2} \quad \dfrac{D4}{\overline{a}_3^1} \quad \dfrac{D4}{\overline{a}_4^1} \quad \dfrac{D4}{\overline{a}_5^1} \quad \ldots}{\{\}}\ y_1y_2 \cup \{a_k^1 \mid k \geq 3\}.$$

§4. Soundness and completeness. RIP contains two independent systems: (Neg) consisting of (Ax) and (Rneg), and
(Pos) consisting of (Ax) and (Rpos).
Sections 4.1, 4.2 show that each system is refutationally complete on its own. The unexpected, paraconsistent features of their combination are described in §5. Notation identifies often one element set with the element, so that a denotes the OR-clause $\{a\}$, while $\overline{a}$ the NAND-clause $\{\overline{a}\}$. Γ, A denotes $\Gamma \cup \{A\}$.

4.1. The system (Neg). The following lemma gives auxiliary results about the deductive closure $Neg(\Gamma)$ of a theory Γ extended with unary clauses.

LEMMA 4.1. *For every* Γ *and* $A \subseteq G$:
1. $Neg(\Gamma \cup A^-) = Neg(\Gamma \cup \{P \setminus B \mid P \in \text{OR}, B \subseteq A\}) \cup A^-$,
2. *for finite* $A : Neg(\Gamma \cup A^+) \supseteq Neg(\Gamma) \cup \{\overline{X \setminus B} \mid \overline{X} \in Neg(\Gamma), B \subseteq A\} \cup A^+$,
   for every $A : Neg(\Gamma \cup A^+) \subseteq Neg(\Gamma) \cup \{\overline{X \setminus B} \mid \overline{X} \in Neg(\Gamma), B \subseteq A\} \cup A^+$.

PROOF. 1. For $\subseteq$, any application of (Rneg) using B^-, for $B \subseteq A$, has a counterpart in the RHS:
$$B \cup \{a_i \mid i \in I\}\ \dfrac{B^- \quad \{\overline{a_iA_i} \mid i \in I\}}{\overline{\bigcup A_i}} \qquad\qquad \dfrac{\{\overline{a_iA_i} \mid i \in I\}}{\overline{\bigcup A_i}}\ \{a_i \mid i \in I\}$$
For $\supseteq$, conclusion of any application as in RHS with $P \setminus B \in \text{OR}$, follows in LHS by a corresponding application with $P \in \text{OR}$ and B^- added to the premises.

2. Since each $B \subseteq A$ is finite, $\supseteq$ follows by a finite number of applications of (Rneg) to $X \in Neg(\Gamma)$ and, successively, each $b \in B$. $\subseteq$ holds for every A since RHS is closed under (Neg). Explicitly, for some index sets $i \in I, k \in K \subseteq J$, with $\overline{X_k x_k B_k} \in Neg(\Gamma)$, $\overline{N_i n_i} \in Neg(\Gamma)$, $B = \bigcup_{k \in K} B_k \subseteq A$ and $N_i n_i \cap A = \varnothing$, the conclusion $\overline{C} = \overline{\bigcup_{i \in I} N_i \cup \bigcup_{k \in K} X_k}$ of

$$\frac{\{\overline{N_i n_i} \mid i \in I\} \quad \{\overline{X_k x_k} \mid k \in K\}}{\overline{\bigcup_{i \in I} N_i \cup \bigcup_{k \in K} X_k}} \ \{n_i \mid i \in I\} \cup \{x_k \mid k \in K\} \in \text{OR}$$

is already in RHS. Namely, $Neg(\Gamma)$ contains the conclusion $\overline{D}$ of the derivation

$$\frac{\{\overline{N_i n_i} \mid i \in I\} \quad \{\overline{X_k x_k B_k} \mid k \in K\}}{\overline{D} = \overline{\bigcup_{i \in I} N_i \cup \bigcup_{k \in K} X_k B_k}} \ \{n_i \mid i \in I\} \cup \{x_k \mid k \in K\} \in \text{OR}$$

and $\overline{C} = \overline{D \setminus B} \in RHS$, since $B \subseteq A$. As $Neg(\Gamma \cup A^+)$ is the smallest set containing $\Gamma \cup A^+$ and closed under (Neg), its inclusion in RHS follows. $\dashv$

We list some consequences of the above lemma relevant for further use, with point 2 being crucial in the proof of completeness.

LEMMA 4.2. *For every* Γ

1. *for finite* $A : \Gamma \cup A^+ \vdash_{Neg} \{\} \Leftarrow \Gamma \vdash_{Neg} \{\} \vee \exists B \subseteq A : \Gamma \vdash_{Neg} \overline{B}$,
   for every $A : \Gamma \cup A^+ \vdash_{Neg} \{\} \Rightarrow \Gamma \vdash_{Neg} \{\} \vee \exists B \subseteq A : \Gamma \vdash_{Neg} \overline{B}$,
2. *for every* $P \subseteq G : (\forall c \in P : \Gamma, c \vdash_{Neg} \{\}) \Rightarrow \Gamma, P \vdash_{Neg} \{\}$,
3. $\Gamma \vdash_{Neg} \{\} \Leftrightarrow \exists K \in \text{OR}\ \forall k \in K : \Gamma \vdash_{Neg} \overline{k}$.

PROOF. 1. ($\Leftarrow$) is obvious since each $B \subseteq A$ is finite. ($\Rightarrow$) If $\{\} \in Neg(\Gamma \cup A^+) \setminus Neg(\Gamma)$ then, by Lemma 4.1.2, for some $B \subseteq A$ and $\overline{X} \in Neg(\Gamma) : \{\} = \overline{X \setminus B}$, i.e., $\overline{X} = \overline{B}$.

2. By point 1, the assumption implies $\Gamma \vdash_{Neg} \{\} \vee \forall c \in P : \Gamma \vdash_{Neg} \overline{c}$. In the latter case, one application of (Rneg) to P and all $\overline{c}, c \in P$, gives $\{\}$.

3. ($\Leftarrow$) is obvious, while ($\Rightarrow$) follows since any derivation of $\{\}$ must end with:

$$\frac{\{\Gamma \vdash_{Neg} \overline{k}_i \mid i \in I\}}{\Gamma \vdash_{Neg} \{\}} \ \{k_i \mid i \in I\} \in \text{OR}. \qquad \dashv$$

(Neg) is sound (also for partial structures) and refutationally complete (for total, classical semantics) for C-F theories.

THEOREM 4.3. *For every* $C \subseteq G$ *and*

1. *for every* $\Gamma : \Gamma \vdash_{Neg} \overline{C} \Rightarrow \Gamma \models \overline{C}$,
2. *for* C-F $\Gamma : Mod(\Gamma) = \varnothing \Rightarrow \Gamma \vdash_{Neg} \{\}$,
3. *for* C-F $\Gamma : \Gamma \models \overline{C} \Leftrightarrow \Gamma \cup C^+ \vdash_{Neg} \{\}$.

PROOF. 1. (Ax) is obviously sound, and so is (Rneg) – for every partial structure $\langle P, N \rangle$: when $P \cap \{a_i \mid i \in I\} \neq \varnothing$, then some $a_{i_0} \in P$, and so $A_{i_0} \cap N \neq \varnothing$, since for every $i : a_i A_i \cap N \neq \varnothing$. Hence, $\bigcup_i A_i \cap N \neq \varnothing$.

2. Enumerate $\text{OR} = \{P_1, P_2, ...\}$ and let $\Gamma_i = \text{NAND} \cup \{P_j \mid j > i\}$. Assume $\Gamma \not\vdash_{Neg} \{\}$. Then, by 4.2.2, there is a $c_1 \in P_1 : c_1, \Gamma_1 \not\vdash_{Neg} \{\}$, and this follows by induction for every $i \in \omega : c_1, c_2 ... c_i, \Gamma_i \not\vdash_{Neg} \{\}$. In the ω-limit, for $C_\omega = \{c_i \mid i \in \omega\}$, we obtain $C_\omega^+ \cup \text{NAND} \not\vdash_{Neg} \{\}$, because otherwise $Mod(C_\omega^+ \cup \text{NAND}) = \varnothing$, by soundness of (Neg), i.e., C_ω contains some $N \in \text{NAND}$. As N is finite, so for some $i \in \omega : N \subseteq \{c_1, ..., c_i\}$. But then $c_1, ..., c_i, \Gamma_i \vdash_{Neg} \{\}$. So $C_\omega \in Tr(\text{OR})$ and

$\forall N \in \text{NAND} : N \not\subseteq C_\omega$, i.e., C_ω gives a model of Γ, by Fact 2.5.

3. $\Gamma \models \overline{C} \Leftrightarrow Mod(\Gamma \cup C^+) = \varnothing \overset{1,2}{\Longleftrightarrow} \Gamma \cup C^+ \vdash_{Neg} \{\}$. $\dashv$

Corollary 4.4. *A countable graph* Γ *has a kernel iff*

$$\forall x \in G\ \exists y \in \mathsf{N}(x) : \Gamma \vdash_{Neg} \overline{x} \Rightarrow \Gamma \nvdash_{Neg} \overline{y}.$$

Proof. ($\Rightarrow$) follows from soundness of (Neg). ($\Leftarrow$) If Γ has no model then, by Theorem 4.3, $\Gamma \vdash_{Neg} \{\}$, i.e., for some $\mathsf{N}[x] \in \text{OR} : \Gamma \vdash_{Neg} \overline{z}$ for all $z \in \mathsf{N}[x]$. We thus have $\Gamma \vdash_{Neg} \overline{x}$ and $\forall y \in \mathsf{N}(x) = \mathsf{N}[x] \setminus \{x\} : \Gamma \vdash_{Neg} \overline{y}$. $\dashv$

An adaptation of the completeness proof, yields also the following fact.

Fact 4.5. *A countable* Γ *with all clauses infinite, has a model.*

Proof. Enumerate $\text{OR} = \{P_1, P_2, ...\}$ and $\text{NAND} = \{N_1, N_2, N_3, ...\}$. Using AC, well-order each P_i and N_i, and let $\mu(X)$ denote the least element of X wrt. this well-ordering. Start with:

$n_1 = \mu(N_1)$ and $c_1 = \mu(P_1 \setminus \{n_1\})$,

and then, inductively, given $\{n_1...n_i\}$ and $\{c_1...c_i\}$, let:

$n_{i+1} = \mu(N_{i+1} \setminus \{c_1...c_i\})$ and $c_{i+1} = \mu(P_{i+1} \setminus \{n_1...n_i, n_{i+1}\})$

Since each P_i, N_i is infinite, such a choice is possible for every finite $i \in \omega$.

The entire $C^* = \{c_i \mid i \in \omega\}$ is then a transversal of OR and $N^* = \{n_i \mid i \in \omega\}$ of NAND. Also $N^* \cap C^* = \varnothing$, for every $c_i \in P_i \setminus \{n_1...n_i\}$, so $c_i \neq n_j$ for all $j \leq i$, while for every $k > i : n_k \in N_k \setminus \{...c_i...\}$, so $c_i \neq n_k$. Since for every $N \in \text{NAND} : N \not\subseteq C^*$, so C^* gives a model of Γ by Fact 2.5. $\dashv$

4.2. The system (Pos). The argument for (Pos) follows the one for (Neg).

Lemma 4.6. *For every* Γ *and* $A \subseteq G$*:*

1. $Pos(\Gamma \cup A^-) = Pos(\Gamma) \cup \{X \setminus P \mid X \in Pos(\Gamma), P \subseteq A\} \cup A^-$
2. $Pos(\Gamma \cup A^+) = Pos(\Gamma) \cup A \cup$
$\{X \setminus \bigcup K_i \mid X \in Pos(\Gamma), a_i \in B_i \subseteq A, \forall k \in K_i : \overline{a_i k} \in \text{NAND}\}$.

Proof. $\supseteq$ are obvious. For $\subseteq$ we show that the RHSs are closed under (Pos).
1. The only (Rpos) applications using A^- are of the form $\frac{X \ P^-}{X \setminus P}$, for some $P \subseteq A$, and RHS is clearly closed under such applications. So consider an application with $X, C_j \in Pos(\Gamma)$ and $P, P_j \subseteq A$:

$$(Rpos)\frac{X \setminus P \qquad \{C_j \setminus P_j \mid j \in J\}}{Z = \ (X \setminus P \setminus \{c_j \mid j \in J\}) \cup (\bigcup (C_j \setminus P_j) \setminus C'_j)}\{\overline{c_j c} \mid c \in C'_j\} \subseteq \text{NAND}$$

If all $P, P_j = \varnothing$, then $Z \in Pos(\Gamma)$. Otherwise, the following $W \in Pos(\Gamma)$:

$$(Rpos)\frac{X \qquad \{C_j \mid j \in J\}}{W = \ (X \setminus \{c_j \mid j \in J\}) \cup (\bigcup C_j \setminus C'_j)}\ \{\overline{c_j c} \mid c \in C'_j\} \subseteq \text{NAND}$$

Thus $Z = W \setminus P'$ for some $P' \subseteq P \cup \bigcup P_j \subseteq A$, i.e., $Z \in RHS$, and so $\Gamma \cup A^- \subseteq Pos(RHS) \subseteq RHS$. Since $Pos(\Gamma \cup A^-)$ is the smallest set containing $\Gamma \cup A^-$ and closed under (Pos), it follows that $Pos(\Gamma \cup A^-) \subseteq RHS$.
2. The argument is the same as in 1, with each P, P_j being now some $\bigcup_{a \in B} K_a$, for various $B \subseteq A$ such that $\bigcup_{a \in B}\{\overline{ak} \mid k \in K_a\} \subseteq \text{NAND}$. $\dashv$

Point 3 of the following Lemma is used in the completeness proof.

LEMMA 4.7. *For every* Γ *and* $A \subseteq G$*:*

1. $\Gamma \cup A^- \vdash_{Pos} \{\} \Leftrightarrow \Gamma \vdash_{Pos} \{\} \vee \exists B \subseteq A : \Gamma \vdash_{Pos} B$.
2. $\Gamma, a \vdash_{Pos} \{\} \Leftrightarrow \Gamma \vdash_{Pos} \{\} \vee (\exists K \subseteq G : \Gamma \vdash_{Pos} K \wedge \{\overline{ak} \mid k \in K\} \subseteq \text{NAND})$
3. $(\forall a_i \in A : \Gamma, a_i \vdash_{Pos} \{\}) \Rightarrow \Gamma, A \vdash_{Pos} \{\}$

PROOF. Implications to the left in 1-2 are obvious, while the opposite ones use Lemma 4.6. If $\{\} \in Pos(\Gamma \cup A^-) \setminus Pos(\Gamma)$, then $\{\} = X \setminus A$ for some $X \in Pos(\Gamma)$, by 4.6.1, so $X = B$ for some $B \subseteq A$. Similarly, in 2, $\{\} = X \setminus K$ for some K with $\{\overline{ak} \mid k \in K\} \subseteq \text{NAND}$ by 4.6.2.

3. follows from 2, which then implies that $\forall a_i \in A\ \exists K_i : \Gamma \vdash_{Pos} K_i$ with $\{\overline{a_i k} \mid k \in K_i\} \subseteq \text{NAND}$, so that (Pos) $\dfrac{A\ ;\ \{K_i \mid i \in I\}}{\{\}}$. ⊣

The argument from Theorem 4.3 gives also refutational completeness of (Pos).

THEOREM 4.8. *For every* $C \subseteq G$ *:*

1. For every $\Gamma : \Gamma \models C \Leftarrow \Gamma \vdash_{Pos} C$*,*
2. For C-F $\Gamma : Mod(\Gamma) = \varnothing \Rightarrow \Gamma \vdash_{Pos} \{\}$*,*
3. For C-F $\Gamma : \Gamma \models C \Leftrightarrow \Gamma \cup C^- \vdash_{Pos} \{\}$*.*

PROOF. 1. The rule (Rpos) is sound: for every partial structure $\langle P, N\rangle$ satisfying the premises, either $\forall i \in I : a_i \notin P$, in which case $A \cap P \neq \varnothing$ implies $(A \setminus \{a_i \mid i \in I\}) \cap P \neq \varnothing$, giving that $\langle P, N\rangle$ satisfies the conclusion, or else $\exists i : a_i \in P$. Then also $a_i \notin N$ and hence for all $k \in K_i : k \in N$ and since $B_i K_i \cap P \neq \varnothing$, so $B_i \cap P \neq \varnothing$, i.e., $\langle P, N\rangle$ satisfies the conclusion.
2. Enumerate $\text{OR} = \{P_1, P_2, ...\}$, and let $\Gamma_k = \{P_i \mid k < i < \omega\} \cup \text{NAND}$. If $\Gamma \not\vdash_{Pos} \{\}$, then there is a $c_1 \in P_1 : c_1, \Gamma_1 \not\vdash_{Pos} \{\}$ by 4.7.3. By induction, the same holds for every finite $i : c_1 ... c_i, \Gamma_i \not\vdash_{Pos} \{\}$. In the ω-limit, for $C_\omega = \{c_i \mid i \in \omega\}$, we obtain $C^+_\omega \cup \text{NAND} \not\vdash_{Pos} \{\}$, for otherwise, by soundness of (Pos), $Mod(C^+_\omega \cup \text{NAND}) = \varnothing$, i.e., for some $N \in \text{NAND} : N \subset C_\omega$. Since each $N \in \text{NAND}$ is finite, for some $k \in \omega : N \subseteq \{c_1, ..., c_k\}$. But then $c_1 ... c_k, \Gamma_k \vdash_{Pos} \{\}$. So, $\forall N \in \text{NAND} : N \not\subseteq C_\omega \in Tr(\text{OR})$, i.e C_ω gives a model of Γ, by Fact 2.5.
3. $\Gamma \models C \Leftrightarrow Mod(\Gamma \cup C^-) = \varnothing \overset{1,2}{\Longleftrightarrow} \Gamma \cup C^- \vdash_{Neg} \{\}$. ⊣

4.3. The whole system. Points 1 and 3 of the following corollary witness to the conservativity of **RIP** over each subsystem. Still, it offers a new tool for handling paradox, which arises from point 4.

COROLLARY 4.9. *For* C-F Γ*:*

1. $Mod(\Gamma) = \varnothing \Leftrightarrow \Gamma \vdash_{Neg} \{\} \Leftrightarrow \Gamma \vdash_{Pos} \{\} \Leftrightarrow \Gamma \vdash \{\}$
2. $\Gamma, x \vdash \{\} \Leftrightarrow (\Gamma \vdash \overline{x} \vee \Gamma \vdash \{\})$ *and* $\Gamma, \overline{x} \vdash \{\} \Leftrightarrow (\Gamma \vdash x \vee \Gamma \vdash \{\})$
3. *If* $\Gamma \not\vdash \{\}$ *then* $(\Gamma \vdash \overline{x} \Leftrightarrow \Gamma \vdash_{Neg} \overline{x})$ *and* $(\Gamma \vdash x \Leftrightarrow \Gamma \vdash_{Pos} x)$
4. $\Gamma \vdash \{\} \Leftrightarrow \exists x : \Gamma \vdash x \wedge \Gamma \vdash \overline{x}$ *(denoted* $\Gamma \vdash \bot(x)$*)*

PROOF. 1. The first two equivalences are Theorems 4.3 and 4.8, giving soundness and refutational completeness of the whole system, i.e., the last equivalence.

2. When $\Gamma \not\vdash \{\}$, we have: $\Gamma \not\vdash \overline{x} \Rightarrow \Gamma \not\vdash_{Neg} \overline{x} \overset{4.2.1}{\Longrightarrow} \Gamma, x \not\vdash_{Neg} \{\} \overset{1}{\Leftrightarrow} \Gamma, x \not\vdash \{\}$.
Conversely, if $\Gamma \vdash \overline{x}$ then also $\Gamma, x \vdash \overline{x}$, while $\Gamma, x \vdash x$, so $\Gamma, x \vdash \{\}$.
Similarly, if $\Gamma \not\vdash \{\}$: $\Gamma \not\vdash x \Rightarrow \Gamma \not\vdash_{Pos} x \overset{4.7.1}{\Longrightarrow} \Gamma, \overline{x} \not\vdash_{Pos} \{\} \overset{1}{\Leftrightarrow} \Gamma, \overline{x} \not\vdash \{\}$.
Conversely, if $\Gamma \vdash x$ then also $\Gamma, \overline{x} \vdash x$, while $\Gamma, \overline{x} \vdash \overline{x}$, so $\Gamma, \overline{x} \vdash \{\}$.

3. In both cases, the implication ($\Leftarrow$) is obvious. For ($\Rightarrow$) assume $\Gamma \not\vdash \{\}$:
$\Gamma \not\vdash_{Neg} \overline{x} \overset{4.2.1}{\Longrightarrow} \Gamma, x \not\vdash_{Neg} \{\} \overset{1}{\Leftrightarrow} \Gamma, x \not\vdash \{\} \overset{2}{\Leftrightarrow} \Gamma \not\vdash \overline{x}$.
$\Gamma \not\vdash_{Pos} x \overset{4.7.1}{\Longrightarrow} \Gamma, \overline{x} \not\vdash_{Pos} \{\} \overset{1}{\Leftrightarrow} \Gamma, \overline{x} \not\vdash \{\} \overset{2}{\Leftrightarrow} \Gamma \not\vdash x$.
4. ($\Leftarrow$) follows by a single application of (Rneg) or (Rpos). ($\Rightarrow$) If $\Gamma \vdash \{\}$ then, by 1, also $\Gamma \vdash_{Neg} \{\}$. Hence, by 4.2.3, there is a clause $K \in$ OR such that $\forall k_i \in K : \Gamma \vdash_{Neg} \overline{k}_i$. Choosing then any $k_0 \in K$, an application of (Rpos) yields $\dfrac{K \quad (K \setminus k_0)^-}{k_0}$ witnessing to the claim. ⊣

Provability of both x and $\overline{x}$, not only comes closer to the informal understanding of paradox than does provability of $\{\}$, but enables also its finer treatment. Before describing it in the next section, let us close this one by observing that we can hardly expect any complete and useful extension of the logic to uncountable theories. Various distributivity laws, used typically for this purpose, have namely semantic character, which reduces them to triviality for OR+NAND theories. For instance, Chang's law postulates that, for a language $\mathcal{L}_\kappa$

$\bigvee_{a<\kappa}(\bigwedge_{b<\kappa} x_{ab})$ is an axiom iff $\forall C \in \kappa^\kappa \exists x : \{x, \neg x\} \subset \{x_{aC(a)} \mid a < \kappa\}$,

or, equivalently:

$\bigwedge_{a<\kappa}(\bigvee_{b<\kappa} x_{ab})$ is an axiom iff $\exists C \in \kappa^\kappa \forall x : \{x, \neg x\} \not\subset \{x_{aC(a)} \mid a < \kappa\}$.

The formula on the left corresponds to a set of clauses, while the right-hand side claims the existence of a choice C selecting, for every $a < \kappa$, an element $x_{aC(a)}$ from the a-th clause $\bigvee_{b<\kappa} x_{ab}$, so that the selection from all $< \kappa$ clauses contains no complementary pair $x, \neg x$. In OR+NAND theories, complementary pairs, selected from distinct clauses, correspond to NAND-pairs. We can therefore rewrite this last formulation as:

OR $= \{P_a \mid a < \kappa\}$ is axiomatic iff $\exists C \in Tr(\text{OR}) : \forall N \in$ NAND $: N \not\subset C$.

But this is definition of a model, as in Fact 2.5. Having it as an axiom, to obtain completeness for $\kappa > \omega_1$, makes reasoning unnecessary.

§5. Nonexplosiveness. We now use **RIP** only for direct, not refutational, reasoning, i.e., for $A \subseteq G$, we are asking simply if $\Gamma \vdash \ddot{A}$. Completeness becomes then limited, missing some redundant clauses. (Occurrences of $\ddot{\_}$ are, in a given context, either all positive or all negative.)

COROLLARY 5.1. *For* C-F Γ *and* $A \subseteq G : \Gamma \models \ddot{A} \Leftrightarrow \exists B \subseteq A : \Gamma \vdash \ddot{B}$.

PROOF. If $Mod(\Gamma) = \varnothing$, then $\Gamma \vdash \{\}$ by 4.9.1 and $\{\} \subseteq A$. Conversely, if $\Gamma \vdash \{\}$, then $Mod(\Gamma) = \varnothing$ so $\Gamma \models \ddot{A}$ for every $A \supseteq \{\}$. This special case is the same for both cases below, which are considered assuming $\Gamma \not\vdash \{\}$:
$\ddot{X} = \overline{X}$. $\exists B \subseteq A : \Gamma \vdash \overline{B} \overset{4.3.1}{\Longrightarrow} \Gamma \models \overline{B} \Rightarrow \Gamma \models \overline{A}$, while the opposite: $\Gamma \models \overline{A} \Leftrightarrow Mod(\Gamma \cup A^+) = \varnothing \overset{4.9.1}{\Longleftrightarrow} \Gamma \cup A^+ \vdash_{Neg} \{\} \overset{4.2.1}{\Longrightarrow} \exists B \subseteq A : \Gamma \vdash_{Neg} \overline{B} \Rightarrow \Gamma \vdash \overline{B}$.
$\ddot{X} = X$. $\exists B \subseteq A : \Gamma \vdash B \overset{4.8.1}{\Longrightarrow} \Gamma \models B \Rightarrow \Gamma \models A$, while the opposite: $\Gamma \models A \Leftrightarrow Mod(\Gamma \cup A^-) = \varnothing \overset{4.9.1}{\Longleftrightarrow} \Gamma \cup A^- \vdash_{Pos} \{\} \overset{4.7.1}{\Longleftrightarrow} \exists B \subseteq A : \Gamma \vdash_{Pos} B \Rightarrow \Gamma \vdash B$. ⊣

The resulting logic does not have weakening – hence neither *Ex Falso Quodlibet*. Its nonexplosiveness gives a paraconsistent ability to contain paradox and reason – classically – about the subdiscourse unaffected by it.

Example 5.2. *The closure of* $y \to z \to x$ *contains, besides* $\{\}$*, all literals. Provability of both* x *and* $\overline{x}$*, i.e., the paradox at* x*, pollutes the whole discourse.*

In the discourse $\{yz, \overline{yz}, zxs, \overline{zx}, \overline{zs}, x, \overline{x}, s\}$*, i.e.,* $s \quad y \to z \to x$*, we still have paradox at* x *and* $\{\}$ *is still provable, but neither is* $\overline{y}$ *nor* z*. The closure contains only the literals* $\{x, \overline{x}, s, \overline{z}, y\}$*, showing that* x *is the only problem, which does not affect the rest of the discourse.*

To identify semantic counterpart of this nonexplosiveness, we first register a form of monotonicity of reasoning. For $\Gamma \subseteq \mathcal{P}(Y)$ and $X \subseteq Y$ we denote the result of removing all atoms X from all clauses of Γ (removing also the empty clause, if it appears in the process):

$$\Gamma \setminus\!\!\setminus X = \{C \setminus X \mid C \in \Gamma\} \setminus \{\{\}\}.$$

This operation corresponds roughly to taking the theory of the subgraph induced by $G \setminus X$.[4] We return to this in connection with its application.

Lemma 5.3. *For every* $\Gamma : \Gamma \vdash \ddot{A} \not\subseteq X \Rightarrow \exists B \subseteq A \setminus X : \Gamma \setminus\!\!\setminus X \vdash \ddot{B}$.

Proof. By induction on the well-founded structure of the proof $\Gamma \vdash \ddot{A}$, with axioms introducing $\ddot{\overline{A \setminus X}}$ instead of $\ddot{A}$. Let $\Gamma' = \Gamma \setminus\!\!\setminus X$. If $\Gamma' \vdash \{\}$, the claim follows, so we assume (especially in IH) nonemptiness of all Γ'-provable clauses. For the induction step

$$(Rneg)\ \ \frac{\{\Gamma \vdash \overline{a_i A_i} \mid i \in I\} \quad \Gamma \vdash \{a_i \mid i \in I\}}{\Gamma \vdash \overline{\bigcup_{i \in I} A_i}},$$

where $\bigcup_{i \in I} A_i \setminus X \neq \varnothing$, there are also some $a_i A_i \setminus X \neq \varnothing$ and we consider only these. If for some $i : \Gamma' \vdash \overline{B}_i \subseteq \overline{A_i \setminus X}$, the $\overline{B}_i$ gives the claim. Otherwise, IH gives for every $i : \Gamma' \vdash \overline{a_i B_i}$, where $B_i \subseteq A_i \setminus X$, while for the side premise, $\Gamma' \vdash \{a_i \mid i \in I'\} \subseteq \{a_i \mid i \in I\} \setminus X$. Appplying (Rneg) to the respective $\Gamma' \vdash \overline{a_i B_i}, i \in I'$ yields the claim with $\overline{\bigcup_{i \in I'} B_i} \subseteq \overline{\bigcup_{i \in I} A_i \setminus X}$.

Induction step for the proof ending with (Rpos), where $\forall i \in I : K_i \subseteq A_i$:

$$(Rpos)\ \frac{\Gamma \vdash A \quad \{\Gamma \vdash A_i \mid i \in I\} \quad \{\Gamma \vdash \overline{a_i k} \mid i \in I, k \in K_i\}}{\Gamma \vdash (A \setminus \{a_i \mid i \in I\}) \cup \bigcup_{i \in I}(A_i \setminus K_i)}$$

By IH, $\Gamma' \vdash B \subseteq A \setminus X$ and $\Gamma' \vdash B_i \subseteq A_i \setminus X$, with $B, B_i \neq \{\}$, for all $i \in I$. If all $a_i \in X$, then $\Gamma' \vdash B$ gives the claim. Likewise, if for some $i : K_i \subseteq X$, then $\Gamma' \vdash B_i \subseteq A_i \setminus X \subseteq A_i \setminus K_i$ gives the claim. Otherwise, consider only $J = \{i \in I \mid a_i \notin X\}$. For every $i \in J$:

(1) $\exists k : k \in K_i \cap X$, and then $\Gamma' \vdash \overline{a_i}$ (by IH $\Gamma' \vdash C \subseteq \overline{a_i k \setminus k}$, and $\Gamma' \not\vdash \{\}$), or

(2) $K_i \cap X = \varnothing$ and then either

- (2.a) $\forall k \in K_i : \Gamma' \vdash \overline{k}$, or
- (2.b) $\Gamma' \vdash \overline{a_i}$, or
- (2.c) $K_i = L_i \uplus R_i \wedge L_i \neq \varnothing \wedge \forall k : (k \in L_i \to \Gamma' \vdash \overline{a_i k}) \wedge (k \in R_i \to \Gamma' \vdash \overline{k})$.

If (2.a) holds for an $i \in J$, the claim follows by $\dfrac{\Gamma' \vdash B_i \quad \{\Gamma' \vdash \overline{k} \mid k \in K_i\}}{\Gamma' \vdash B_i \setminus K_i}$.

[4]For $H \subseteq G$, the subgraph of $\langle G, \mathsf{N} \rangle$ induced by H is $\langle H, \mathsf{N}_H \rangle$ with $\mathsf{N}_H = \mathsf{N} \cap (H \times H)$.

Otherwise, for all i satisfying (2.b) or (1), apply first (Rpos) to $\Gamma' \vdash B$ obtaining $\Gamma' \vdash B' = B \setminus \{a_i \mid \Gamma' \vdash \overline{a_i}\}$. There remain i's from (2.c), i.e., $I' = \{i \in J \mid \Gamma' \not\vdash \overline{a_i} \wedge L_i \neq \varnothing\}$:

$$\frac{\begin{array}{c}\vdots\\ \Gamma' \vdash B'\end{array} \quad \left\{\frac{\Gamma' \vdash B_i \quad \{\Gamma' \vdash \overline{k} \mid k \in R_i\}}{\Gamma' \vdash B_i \setminus R_i} \mid i \in I'\right\} \quad \left\{\begin{array}{c}\vdots\\ \Gamma' \vdash \overline{a_i k}\end{array} \mid i \in I', k \in L_i\right\}}{\Gamma' \vdash (B' \setminus \{a_i \mid i \in I'\}) \cup \bigcup_{i \in I'} (B_i \setminus K_i)}$$

The conclusion of this derivation gives the claim. $\dashv$

The condition like $A \not\subseteq X$ is needed because the transition to $\Gamma \setminus\!\!\setminus X$ neither preserves nor reflects provability of $\{\}$. For instance, $\Gamma_1 = \{s, x, \overline{x}\} \vdash \{\}$, but $\Gamma_1 \setminus\!\!\setminus \{x\} = \{s\} \not\vdash \{\}$, while $\Gamma_2 = \{s, xs, \overline{x}, \overline{xs}\} \not\vdash \{\}$, but $\Gamma_2 \setminus\!\!\setminus \{s\} = \{x, \overline{x}\} \vdash \{\}$.

5.1. The paradoxical and the consistent subdiscourses. Turning now to paradoxical discourses, let $\Gamma \vdash \{\}$ and denote:

$G^{\perp} = \{x \in G \mid \Gamma \vdash x \wedge \Gamma \vdash \overline{x}\}$
$\Gamma^{ok} = \Gamma \setminus\!\!\setminus G^{\perp} = \{C \setminus G^{\perp} \mid C \in \Gamma\} \setminus \{\{\}\}$
$G^{ok} = G \setminus G^{\perp} = \bigcup \Gamma^{ok}$.

$G^{\perp}$ contains all statements involved in the paradox and the story ends here when it covers the whole G. But otherwise Γ^{ok} remains consistent alongside $G^{\perp}$.

Fact 5.4. *For* c-f Γ *with* $G^{ok} \neq \varnothing$*:*

1. $\forall \ddot{D} \in \Gamma^{ok} : \Gamma \vdash \ddot{D}$*, and so* $\Gamma^{ok} \vdash \ddot{C} \Rightarrow \Gamma \vdash \ddot{C}$*, for any* $C \subseteq G^{ok}$*.*
2. $\Gamma^{ok} \not\vdash \{\}$*.*
3. $\forall x \in G^{ok} : \Gamma^{ok} \vdash x \Leftrightarrow \Gamma \vdash x$ *and* $\Gamma^{ok} \vdash \overline{x} \Leftrightarrow \Gamma \vdash \overline{x}$*.*
4. $\exists x \in G^{ok} : \Gamma^{ok} \not\vdash \overline{x}$*, hence also* $\Gamma \not\vdash \overline{x}$*.*
5. $\forall x \in G^{ok} : \Gamma^{ok} \not\vdash \overline{x} \Rightarrow \mathsf{N}(x) \cap G^{\perp} = \varnothing$ *(when* Γ *is a graph).*

Proof. 1. $\forall \ddot{D} \in \Gamma^{ok} \setminus \Gamma\ \exists \ddot{C} \in \Gamma : \ddot{D} = C \setminus (\ddot{C} \cap G^{\perp})$. Two cases:

$$\text{(Rpos)}\ \frac{\Gamma \vdash C \quad \{\Gamma \vdash \overline{c} \mid c \in C \cap G^{\perp}\}}{\Gamma \vdash C \setminus (C \cap G^{\perp})} \qquad \frac{\Gamma \vdash \overline{C} \quad \Gamma \vdash c\ (c \in C \cap G^{\perp})}{\Gamma \vdash \overline{C \setminus \{c\}}}\ \text{(Rneg)}$$

$\overline{C} \in$ nand $\subseteq \Gamma$ is finite, so finitely many applications of (Rneg) suffice to get $\overline{D} = \overline{C \setminus (C \cap G^{\perp})} \in \Gamma^{ok}$ in the later case.

2. $\Gamma^{ok} \vdash \{\} \stackrel{4.9.4}{\Longrightarrow} \exists x \in G^{ok} : \Gamma^{ok} \vdash x \wedge \Gamma^{ok} \vdash \overline{x} \stackrel{1}{\Rightarrow} \Gamma \vdash x \wedge \Gamma \vdash \overline{x} \Rightarrow x \notin G^{ok}$.

3. ($\Rightarrow$) follow from point 1, while ($\Leftarrow$) by Lemma 5.3 and point 2.

4. If $\forall x \in G^{ok} : \Gamma^{ok} \vdash \overline{x}$ then $\forall y \in G : \Gamma \vdash \overline{y}$. But then also $\forall y \in G : \Gamma \vdash y$, contradicting $G^{ok} \neq \varnothing$.

5. If $x \in G^{ok}$ has a $y \in \mathsf{N}(x) \cap G^{\perp}$, then $\Gamma \vdash y \Rightarrow \Gamma \vdash \overline{x} \stackrel{3}{\Rightarrow} \Gamma^{ok} \vdash \overline{x}$. $\dashv$

For a graph Γ, Γ^{ok} is almost the theory of its subgraph induced by G^{ok}, except for some differences at its border vertices $brd(G^{ok}) = \{x \in G^{ok} \mid \mathsf{N}(x) \not\subseteq G^{ok}\}$.

Example 5.5. *Consider the following* $\Gamma : y \to z \to x \quad s \rightleftarrows t$*.*

$$\begin{array}{lcl}
\Gamma & = & \{yz, \overline{yz}, zxs, \overline{zx}, \overline{zs}, st, \overline{st}, x, \overline{x}\}\\
G^{\perp} & = & \{x\}\\
G^{ok} & = & \{y, z, s, t\}\\
\Gamma^{ok} & = & \{yz, \overline{yz}, zs, \overline{z}, \overline{zs}, st, \overline{st}\}\\
brd(G^{ok}) & = & \{z\}
\end{array}$$

The theory of the subgraph induced by G^{ok} *is* $\mathcal{T}(\underline{\Gamma}^{ok}) = \{yz, \overline{yz}, zs, \overline{zs}, st, \overline{st}\}$*, while* Γ^{ok} *contains, in addition,* $\overline{z}$*.*

Border vertices enter as such negative clauses into $\Gamma^{ok} = \mathcal{T}(\underline{\Gamma}^{ok}) \cup (brd(G^{ok}))^-$, so we can view Γ^{ok} as the subgraph $\underline{\Gamma}^{ok}$ induced by G^{ok}, with a new loop at each border vertex. It is not paradoxical, Fact 5.4.2, and its models are kernels of $\underline{\Gamma}^{ok}$ excluding border vertices: $Mod(\Gamma^{ok}) = \{K \in Ker(\underline{\Gamma}^{ok}) \mid brd(G^{ok}) \subseteq \mathsf{N}^{\smile}(K)\}$.[5] In the above example, $\underline{\Gamma}^{ok}$ has two kernels $\{t, z\}$ and $\{s, y\}$, but only the latter gives a model of Γ^{ok}, which requires $\overline{z}$.

The relation between paradoxical graph Γ and Γ^{ok} can be specified further in more semantic terms. Models of Γ^{ok} are namely semikernels of Γ. A *semikernel* is a subset $L \subseteq G$ such that $\mathsf{N}(L) \subseteq \mathsf{N}^{\smile}(L) \subseteq G \setminus L$. It determines the subdiscourse induced by $L \cup \mathsf{N}^{\smile}(L)$, with the partial structure $\alpha_L^{\smile} = \langle L, \mathsf{N}^{\smile}(L)\rangle$.[6] Actually, L is a semikernel iff $\alpha_L^{\smile}$ satisfies both conditions of (2.3), [3], so this subdiscourse, when torn apart from Γ, does not involve paradox. For instance, $\Theta_2 : \circlearrowright l \to t \rightleftarrows s \leftarrow r \circlearrowleft$ has no kernel (and $\Theta_2^{ok} = \varnothing$), but $\{t\}$ and $\{s\}$ are semikernels, giving partial structures $\alpha_{\{t\}}^{\smile} = \langle\{t\}, \{l, s\}\rangle$ and $\alpha_{\{s\}}^{\smile} = \langle\{s\}, \{r, t\}\rangle$ satisfying (2.3). Semikernels provide thus the possibility of ignoring part of the context and were used in [3] as the semantics of nonparadoxical subdiscourses. $Mod(\Gamma^{ok})$ specialize this general concept. ($SK(\Gamma)$ denotes all semikernels of Γ.)

Fact 5.6. *For a countable graph* Γ*:* $Mod(\Gamma^{ok}) \subseteq SK(\Gamma)$.

Proof. Assume $K \in Mod(\Gamma^{ok})$. For every $k \in K$, $\Gamma^{ok} \nvdash \overline{k}$ by soundness, so by 5.4.5, $\mathsf{N}(k) \subseteq G^{ok}$, and hence $\mathsf{N}(k) = \mathsf{N}_{\underline{\Gamma}^{ok}}(k)$, since $\underline{\Gamma}^{ok}$ is subgraph of Γ induced by G^{ok}. $K \in Ker(\underline{\Gamma}^{ok})$ gives the first inclusion and the second equality: $\mathsf{N}(K) = \mathsf{N}_{\underline{\Gamma}^{ok}}(K) \subseteq G^{ok} \setminus K = \mathsf{N}^{\smile}_{\underline{\Gamma}^{ok}}(K) \subseteq \mathsf{N}^{\smile}(K)$. We also have $\mathsf{N}^{\smile}(K) \subseteq G \setminus K$, for K is independent in Γ, being independent in the induced subgraph $\underline{\Gamma}^{ok}$. Thus $\mathsf{N}(K) \subseteq \mathsf{N}^{\smile}(K) \subseteq G \setminus K$, i.e., $K \in SK(\Gamma)$. ⊣

The soundness arguments in Theorems 4.3 and 4.8 apply to the partial structures and not only to the classical ones. For a graph Γ, a partial structure $\langle P, N\rangle \models \Gamma$ is in fact a classical model, which exists if $\Gamma = \Gamma^{ok}$. But when Γ has no model, yet has a subdiscourse $\Gamma^{ok} \nvdash \{\}$, the models of Γ^{ok}, induced from some semikernels of Γ, are partial structures for Γ. Semantic situation is one of the three kinds, depending on the relation between G^{ok} and G:

	$\Gamma \vdash \{\}$	$\Gamma \vdash \bot(x)$	
$G^{ok} = G$	*no*	*no* x	$Mod(\Gamma^{ok}) = Mod(\Gamma) \neq \varnothing$
$\varnothing \neq G^{ok} \subset G$	*yes*	$x \in G^{\bot}$	$Mod(\Gamma^{ok}) \neq \varnothing = Mod(\Gamma)$
$G^{ok} = \varnothing$	*yes*	*all* x	$Mod(\Gamma^{ok}) = \varnothing = Mod(\Gamma)$

The semantics $Mod(\Gamma^{ok})$ – of Γ – explains the nonexplosive behavior: reasoning from Γ is sound also for these partial structures. Besides contrarieties $\bot(x)$, provable when $G^{\bot} \neq \varnothing$, **RIP** proves neither simply facts true in all kernels of Γ (as does classical logic), nor simply facts implied by all its semikernels (as does **Ł3**, [3]), but facts true in maximal semikernels which are not infected by paradox, namely, $Mod(\Gamma^{ok}) \subseteq Ker(\underline{\Gamma}^{ok}) \cap SK(\Gamma)$. For literals (in countable graphs), this is Fact 5.4.3, while the following implies the general case for arbitrary graphs.

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[5] This makes sense as $\forall b \in brd(\underline{\Gamma}^{ok}) : b \notin sinks(\underline{\Gamma}^{ok})$, since $\varnothing \neq \mathsf{N}(b) \subseteq G^{\bot} \overset{5.9}{\Rightarrow} b \in G^{\bot}$.

[6] It coincides with $\alpha_L = (L, G \setminus L)$ only when L is a kernel. Subdiscourse corresponds to an induced subgraph, rather than to a subtheory.

THEOREM 5.7. *For any* Γ, *denote* $Th(\Gamma)|_{G^{ok}} = \{\ddot{C} \subseteq G^{ok} \mid \Gamma \vdash \ddot{C}\}$*:*
1. $Mod(\Gamma^{ok}) \subseteq Mod(Th(\Gamma)|_{G^{ok}})$ *– for every* Γ*;*
2. $Mod(\Gamma^{ok}) \supseteq Mod(Th(\Gamma)|_{G^{ok}})$ *– for* Γ *with all* $N \in$ NAND *finite.*

PROOF. The nontrivial case is when $\varnothing \neq G^{ok} \neq G$. (1) If $\Gamma \vdash \ddot{C} \subseteq G^{ok}$ then, by Lemma 5.3, $\Gamma^{ok} \vdash \ddot{B} \subseteq \ddot{C}$ ($B \neq \{\}$ since $G^{ok} \neq \varnothing$). For every $M \in Mod(\Gamma^{ok}) : M \models \ddot{B}$, so $M \models \ddot{C}$, i.e., $M \in Mod(Th(\Gamma)|_{G^{ok}})$. (2) follows since $\Gamma^{ok} \subseteq Th(\Gamma)|_{G^{ok}}$ by Fact 5.4.1 (which does not require countable OR.) $\dashv$

5.2. Propagation of paradox. Paradox need not pollute the whole discourse, but it spreads to $G^{\perp}$ and we specify closer the pattern of this spreading.

FACT 5.8. *For any* x *in any graph* $\Gamma : \Gamma \vdash \perp(x) \Rightarrow \forall y \in \mathsf{N}^*(x) : \Gamma \vdash \perp(y)$.

PROOF. $\Gamma \vdash x$ gives the side formula for obtaining $\forall y \in \mathsf{N}(x) : \Gamma \vdash \overline{y}$, which then, together with $\Gamma \vdash \overline{x}$, yield $\forall y \in \mathsf{N}(x) : \Gamma \vdash y$:

(Rneg) $\dfrac{\overline{xy}}{\overline{y}}\ x$ and (Rpos) $\dfrac{\mathsf{N}[x]}{y}\ (\mathsf{N}[x] \setminus \{y\})^{-}$.

Induction gives this for all $y \in \mathsf{N}^n(x)$, for all $n \in \mathbb{N}$, i.e., for all $y \in \mathsf{N}^*(x)$. $\dashv$

So, $\mathsf{N}(G^{\perp}) \subseteq G^{\perp}$ and, dually, $\mathsf{N}^{\smile}(G^{ok}) \subseteq G^{ok}$. This may seem surprising, since reading a path from x to y as x "referring to" or "depending on" y, a paradox pollutes thus everything on which it depends. For instance, in "This statement is false and the sun is a star", i.e., $f \to y \to s$, f "refers to" the sink s. One could say: since s is true (y is false and) f is paradoxical. But this paradox spreads then from f to y and s, neither of which "depends" on it. All literals are provable and the true fact s is also provably false. Contributing to the occurrence of a paradox, which "depends" on it, it is a part of the paradoxical whole.[7]

Paradox can also spread upwards, along $\mathsf{N}^{\smile}$, as in $x \leftarrow z$, where provability of $\perp(x)$ leads to provability of $\perp(z)$. But such upward propagation can be interrupted. In Example 5.5, $G^{ok} = \{y, z, s, t\}$ – both z and y "depend" on the paradox at x, but are not affected by it.

A sufficient condition for an upward propagation of paradox is that all paths from a given statement reach, eventually, a paradox. A *complete path* is a path (i.e., $\pi \in G^I$ with $I \in \omega^+$ and $\pi_{i+1} \in \mathsf{N}(\pi_i)$ for all $i + 1 \in I$) which is infinite or terminates with a sink. $paths(x)$ denotes all paths starting from x.

FACT 5.9. *For an* x *in any graph* Γ*, if every complete* $\pi \in paths(x)$ *contains a paradoxical* π_i*, i.e.,* $\Gamma \vdash \perp(\pi_i)$*, then* $\Gamma \vdash \perp(x)$.

PROOF. Assume x is as stated and $\Gamma \not\vdash \perp(x)$. For every complete $\pi \in paths(x)$, let $x_\pi \in \pi$ be the first vertex on π for which $\Gamma \vdash \perp(x_\pi)$ and $X_\perp = \{x_\pi \mid \pi \in paths(x)\}$. Then $\forall z \in X_0 = \big(\mathsf{N}^*(x) \cap (\mathsf{N}^{\smile})^*(X_\perp)\big) \setminus X_\perp : \Gamma \not\vdash \perp(z)$. The claim is that $\exists z \in X_0 : \mathsf{N}(z) \subseteq X_\perp$. For if not, i.e., $\forall z \in X_0 \exists y \in \mathsf{N}(z) \setminus X_\perp$, then let z_0 be any such and $z_1 \in \mathsf{N}(z_0) \setminus X_\perp$. Given z_i we can choose $z_{i+1} \in \mathsf{N}(z_i) \setminus X_\perp$, obtaining an infinite path from x to (z_0 and then) $\langle z_0, z_1, z_2, ...\rangle$ with no element $\perp(z_i)$, contrary to the assumption. So, a claimed z exists. But since $\mathsf{N}(z) \subseteq X_\perp$, so $\Gamma \vdash \perp(z)$, contradicting $\Gamma \not\vdash \perp(z)$. $\dashv$

[7]This is not to suggest that "The sun is a star" is paradoxical but only that combined with the contingen liar as above, it gives the paradoxical whole. Like consistency, paradox is genuinely holistic. To "repair" it, removing the loop at f is as good as removing s.

Γ from Example 5.5 illustrates thus the only possibility of preventing the propagation of paradox upwards by some path which, exiting from a border vertex, like $z \in brd(\Gamma^{ok})$, meets no paradox and forces z to be false.

§6. Concluding remark. Like in logics with internal truth-predicate, paradox formulated in GNF becomes a special case of inconsistency: a discourse is paradoxical when the T-schemata of its statements, expressed in GNF, are inconsistent. The graphical representation gives a precise grasp of vicious circularities. It confirms, for instance, the intuition that for obtaining a finitary paradox, negative self-reference is necessary (and not only sufficient): according to Richardson's theorem, [4], a finitary graph without odd cycle has a kernel.

Even if some satisfactory logical language, adopting paradox, becomes agreed upon, it will hardly remove the need to identify occurrences of paradox by diagnosing its general patterns and by detailed analysis of the actual cases. For classical logic, kernel theory provides a rich source of such patterns, explored initially in [2, 3]. The analysis enabled by **RIP** can, besides diagnosing paradox, identify the nonparadoxical subdiscourse and its classical concequences, which are not affected by the surrounding inconsistency. This paraconsistent effect is obtained by nonrefutational use of hyper-resolution, which deviates from classical reasoning only by the exclusion of weakening.

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