# Compositional Homomorphisms of Relational Structures (Modeled as Multialgebras)

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**Abstract.** The paper attempts a systematic study of homomorphisms of relational structures. Such structures are modeled as multialgebras (i.e., relation is represented as a set-valued function). The first, main, result is that, under reasonable restrictions on the form of the definition of homomorphism, there are exactly nine compositional homomorphisms of multialgebras. Then the comparison of the obtained categories with respect to the existence of finite limits and co-limits reveals two of them to be finitely complete and co-complete. Without claiming that compositionality and categorical properties are the only possible criteria for selecting a definition of homomorphism, we nevertheless suggest that, for many purposes, these criteria actually *might* be acceptable. For such cases, the paper gives an overview of the available alternatives and a clear indication of their advantages and disadvantages.

## 1 Background and motivation

In the study of universal algebra, the central place occupies the pair of "dual" notions of congruence and homomorphism: every congruence on an algebra induces a homomorphism into a quotient and every homomorphism induces a congruence on the source algebra. Categorical approach attempts to express *all* (internal) properties of algebras in (external) terms of homomorphisms. When passing to relational structures or power set structures, however, the close correspondence of these internal and external aspects seems to get lost.

The most common, and natural, generalisation of the definition of homomorphism to relational structures says:

**Definition 1.1** *A set function $\phi : \underline{A} \to \underline{B}$,*[1] *where both sets are equipped with respective relations $R^A \subseteq \underline{A}^n$ and $R^B \subseteq \underline{B}^n$, is a (weak) homomorphism iff*

$$\langle x_1 ... x_n \rangle \in R^A \ \Rightarrow \langle \phi(x_1) ... \phi(x_n) \rangle \in R^B$$

* The first author gratefully acknowledges the financial support from the Norwegian Research Council.

[1] Underlying sets will be used to indicate the "bare, unstructured sets" as opposed to power sets or other sets with structure. For the moment, one may ignorte this notational convention.

With this definition *any* equivalence on $\underline{A}$ gives rise to a weak homomorphism and, conversely, a weak homomorphism induces, in general, only an equivalence relation on $\underline{A}$. Hence this homomorphism does not capture the notion of congruence and this is just one example of an internal property of relational structures that cannot be accounted for by relational homomorphisms (in various variants). Probably for this reason, the early literature on homomorphisms of relations is extremely meagre [22, 26] and most work on relations concerns the study of relation algebras, various relational operators and their axiomatizations. Although in recent years several authors begun studying relational structures and their homomorphisms in various contexts, a general treatement of relational homomorphisms is still missing. This growing interest is reflected in numerous suggestions on how the definition of such a homomorphism could be specialized to obtain a more useful notion. This issue is our main objective.

In a more concise, relational notation, definition 1.1 is written as $R^A;\phi \subseteq \phi;R^B$. This somehow presupposes that $R$ is a binary relation, since composition $\_;\_$ has a standard definition only for binary relations. There seems to be no generally accepted definition of composition of relations of arbitrary arities. In the following we will compose arbitrary relations (within the structures), like $R$ above, with binary relations (obtained from homomorphisms between the structures), according to the following definition.

**Definition 1.2** *The composition of relations $R^A \subseteq \underline{A}^{n+1}$, resp. $R^B \subseteq \underline{B}^{n+1}$, with a binary relation $\phi \subseteq \underline{A} \times \underline{B}$ as a relation on $\underline{A}^n \times \underline{B}$, is given by:*

$$\langle a_1 ... a_n, b\rangle \in R^A;\phi \Leftrightarrow \exists a \in \underline{A} : \langle a_1 ... a_n, a\rangle \in R^A \ \wedge \ \langle a, b\rangle \in \phi$$
$$\langle a_1 ... a_n, b\rangle \in \phi;R^B \Leftrightarrow \exists b_1 ... b_n \in \underline{B} : \langle b_1 ... b_n, b\rangle \in R^B \ \wedge \ \langle a_i, b_i\rangle \in \phi$$

This definition is certainly not the only possible one – [28, 8] contain more general suggestions. The reason for this choice is our intension to treat relations in an algebraic way. It allows us to view relations as set-valued functions and turns relational structures into algebraic ones (*algebras of complexes* from [15, 16]). In particular, it admits composition of relations of arbitrary arities analogous to composition of functions.

Now, table 1 presents a sample of proposed definitions of relational homomorphisms gathered from [22, 9, 7, 20, 25, 3, 23, 24]. It uses binary relations but with the above definition 1.2 it may be used for relations $R$ of arbitrary arity. The names are by no means standard and taken from the articles introducing the respective definitions.

This paper is an attempt to bring at least some order into this situation which we experience as rather unsatisfactory. Given the combinatorial possibilities of defining homomorphisms of relational structures, a complete classification seems hardly possible. Even the very issue of the "criteria of usefulness", depending on the intended applications, may be debatable. We hope that a more algebraic perspective may bring at least some clarification. Instead of listing and defending new definitions, we have chosen the compositionality of homomorphisms and the elementary properties of the resulting categories as the basis for comparison.

| homomorphism $\phi$ | | relational def. | logical def. $\forall x, y$ : |
|---|---|---|---|
| 1. | weak | $\phi^-; R^A; \phi \subseteq R^B$ | $R^A(x,y) \Rightarrow R^B(\phi(x),\phi(y))$ |
| 2. | loose | $R^A; \phi \subseteq \phi; R^B$ | 1. |
| 3. | full | $\phi^-; R^A; \phi = \phi^-; \phi; R^B; \phi^-; \phi$ | $\exists x', y' : R^A(x',y') \Leftrightarrow R^B(\phi(x),\phi(y))$ |
| 4. | 'strong$'$ | $\phi^-; R^A; \phi \supseteq \phi^-; \phi; R^B; \phi^-; \phi$ | $\exists x', y' : R^A(x',y') \Leftarrow R^B(\phi(x),\phi(y))$ |
| 5. | outdegree | $R^A; \phi = \phi; R^B; \phi^-; \phi$ | $\exists x' : R^A(x',y) \Leftrightarrow R^B(\phi(x),\phi(y))$ |
| 6. | indegree | $\phi^-; R^A = \phi^-; \phi; R^B; \phi^-$ | $\exists y' : R^A(x,y') \Leftrightarrow R^B(\phi(x),\phi(y))$ |
| 7. | 'very strong$'$ | $\phi; \phi^-; R^A; \phi \supseteq \phi; R^B$ | $\exists x', y' : R^A(x',y') \Leftarrow R^B(\phi(x),y)$ |
| 8. | regular | 5. & 6. | 5. & 6. |
| 9. | closed | $R^A; \phi \supseteq \phi; R^B$ | $\exists y' : R^A(x,y') \Leftarrow R^B(\phi(x),y)$ |
| 10. | strong | $R^A = \phi; R^B; \phi^-$ | $R^A(x,y) \Leftrightarrow R^B(\phi(x),\phi(y))$ |
| 11. | tight | $R^A; \phi = \phi; R^B$ | 2. & 9. |

– primed symbol $z'$ denotes some element such that $\phi(z') = \phi(z)$ (in 7 and 9 a $y' : \phi(y') = y$)
– for $\phi \subseteq \underline{A} \times \underline{B}$, $\phi^-$ denotes the inverse $\langle b, a\rangle \in \phi^- \Leftrightarrow \langle a, b\rangle \in \phi$

**Table 1.** Some definitions of relational homomorphisms

Section 2 introduces mulitalgebras as a possible way of representing relations and motivates this choice. Section 3 addresses the question of composition of homomorphisms: 3.1 gives a characterization of homomorphisms which are closed under composition – in fact, most of the suggested defintions, like most of those in table 1, do *not* enjoy this property which we believe is crucial. The proof, included here, is a significant improvement of the original proof of this result from [29]. Subsection 3.2 characterizes the equivalences associated with various compositional homomorphisms. Section 4 summarizes the results on (finite) completeness and co-completeness of the obtained categories.

Except for the statement of the compositionality theorem, the results concerning the three "inner" categories were published earlier in [29, 30]. This paper is the first complete overview with the results for the remaining six categories from [11].

**Preliminary definitions and notation** A *relational signature* $\Sigma$ is a pair $\langle \mathcal{S}, \mathcal{R}\rangle$ where $\mathcal{S}$ is a set (of sort symbols) and $\mathcal{R}$ is a set of relation symbols with given arities (also called *type* of relation), written $[R_i : s_1 \times ... \times s_n] \in \mathcal{R}$. Similarly, *an algebraic signature* is a pair $\langle \mathcal{S}, \mathcal{F}\rangle$ where $\mathcal{F}$ is a set of function symbols with associated sorts and arities, written $[f : s_1 \times ... \times s_n \to s] \in \mathcal{F}$. A *relational structure* over a signature $\Sigma = \langle \mathcal{S}, \mathcal{R}\rangle$ is a pair $A = \langle |A|, \mathcal{R}^A\rangle$, where $|A|$ is an $\mathcal{S}$-sorted set called the *carrier* and $\mathcal{R}^A$ is a set of relations, such that for each $[R_i : s_1 \times ... \times s_n] \in \mathcal{R}$ there is $R_i^A \subseteq |A|_{s_1} \times ... \times |A|_{s_n}$. An *algebra* over a signature $\Sigma = \langle \mathcal{S}, \mathcal{F}\rangle$ is again a pair $A = \langle |A|, \mathcal{F}^A\rangle$, where $|A|$ is an $\mathcal{S}$-sorted set called the *carrier*, and $\mathcal{F}^A$ is a set of functions, with a function $f^A : |A|_{s_1} \times ... \times |A|_{s_n} \to |A|_s$ for each $[f : s_1 \times ... \times s_n \to s] \in \mathcal{F}$.

In order to reduce the complexity of notation, we shall limit ourselves to single sorted structures and algebras ($\mathcal{S}$ has only one element) claiming that the results carry over to a multi-sorted case.

We will study algebras over carriers being power sets. For such a structure $A$ with $|A| = \wp(\underline{A})$, the set $\underline{A}$ will be called the *underlying set*. Given a function $f : \underline{A} \to \underline{B}$, we will often use additive pointwise extension without making it explicit in the notation – for any $X \subset \underline{A}$, we write $f(X)$ meaning $\bigcup_{x \in X} f(x)$. Also, we do not make explicit the distinction between elements and one-element sets – if $A = \wp(\underline{A})$ and $a \in \underline{A}$, we write $a \in A$ meaning $\{a\} \in A$. Homomorphisms of multialgebras map underlying sets to underlying sets, just as the homomorphisms of relational structures do.

Composition is written in diagrammatic order $f;g$ for $g(f(\_))$. For a binary relation/function $\phi$, $\phi^-$ denotes its inverse $= \{\langle y, x\rangle : \langle x, y\rangle \in \phi\}$.

## 2 Multialgebras

Our interest in relational structures originates from earlier studies of multialgebras [4, 5, 10, 13, 17, 21, 27, 30, 32] which provide means for modeling nondeterminism in the context of algebraic specification of abstract data types [13, 17–19, 31–33]. Multilagebras can be viewed as relational structures with a specific composition of relations of arbitrary arities. According to definition 1.2, relations are viewed as set-valued functions where the last, $n$-th argument corresponds to an element of the result set obtained by applying the function to the first $n-1$ arguments. This view appears in [26], was elaborated in [15, 16], then in [27] and re-emerged in recent years in the algebraic approaches to nondeterminism. It is based on the simple observation that any (set-valued) operation $f : A_1 \times ... \times A_n \to \wp(A)$ determines a relation $R_f \subseteq A_1 \times ... \times A_n \times A$ and vice versa, via the isomorphism:

$$A_1 \times ... \times A_n \to \wp(A) \;\simeq\; \wp(A_1 \times ... \times A_n \times A) \tag{2.1}$$

Based on this fact, [15, 14] introduced the concept of *algebra of complexes* which we call *multialgebras* – with one proviso: the carrier of a multialgebra is a power set but Boolean operations are not part of the multialgebraic signature. This seemingly negligible differece turns out to have significant consequences, since signatures determine homomorphisms.

**Definition 2.1** *Given a signature $\Sigma = \langle \mathcal{S}, \mathcal{F}\rangle$, a $\Sigma$-multialgebra $M$ is given by:*
- *a carrier $|M| = \{|M|_s\}_{s\in\mathcal{S}}$, where for each $s \in \mathcal{S}$, $|M|_s = \wp(\underline{M}_s)$ of some* underlying set *$\underline{M}_s$, with the obvious embedding $\underline{M}_s \hookrightarrow \wp(\underline{M}_s)$;*
- *a function $f^M : \underline{M}_{s_1} \times ... \times \underline{M}_{s_n} \to \wp(\underline{M}_s)$ for each $f : s_1 \times ... \times s_n \to s \in \mathcal{F}$, with composition defined through additive extension to sets, i.e. $f^M(X_1, ..., X_n) = \bigcup_{x_i \in X_i} f^M(x_1, ..., x_n)$.*

Multialgebras are "partial" in the sense that operations may return empty set of values. By the pointwise extension of operations, they are strict in all arguments. Notice also that we allow empty carriers, i.e. $\wp(\emptyset) = \{\emptyset\}$ is a possible carrier of a multialgebra.

One consequence of modeling relations by set-valued functions is that possible requirement on the definitions of homomorphisms to be invariant under permutation of variables becomes little (if at all) relevant. Such a requirement is not satisfied, for instance, by homomorphisms 5., 6., 7., 9. and 11. in the table 1 so, using this criterion, one would claim that these definitions are not "appropriate". Functions, on the other hand, do have ordered arguments and, above all, distinguish one relational argument as the result. Thus, the definitions of homomorphisms are indeed invariant under permutation of the argument variables, but there seems to be no reason to require such an invariance under permutation of (some) argument variable(s) with the result.

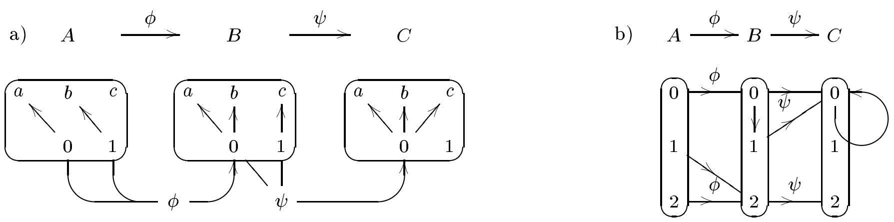
Although multialgebras introduce some structure not present within the corresponding relational structures, the isomorphism (2.1) allows one to convert any multialgebra to the corresponding relational structure and vice versa. Morover, any definition of a homomorphism of multialgebras can be transfered to the relational structures and vice versa, although this may require forgetting/introducing the distinction between the argument and result variables. We now turn to multialgebraic homomorphisms but, for convenience, we will mostly use relational notation.

## 3 Compositionality of multialgebraic homomorphisms

Theorem 3.5, which is the main result of this section, gives an exhaustive characterization of compositional definitions. We begin by giving a counter-example for compositionality of one of the homomorphisms from table 1.

Full homomorphisms were considered in [22, 20] as *the* homomorphisms between relations. In a more special form, they also appear in the study of partial algebras [6]. In the context of partial algebras, it is known that these homomorphisms do not compose. But it is not clear if all the authors were aware of this fact in the general setting.

**Example 3.1** *Let $A, B, C$ be structures with one relation $R$ – a) presents a many-sorted counter-example and b) the single-sorted case.*




*Both $\phi$ and $\psi$ are full homomorphisms. However, due to non-surjectivity of $\phi$, the composition $\phi;\psi$ is not full. Although $\phi;\psi(0) = 0$ and $\langle 0, c\rangle$, resp. $\langle 0, 0\rangle \in R^C$ there is no element $x \in A$ in the pre-image of $c$, resp. $0$ for which we would have $\langle 0, x\rangle \in R^A$.*

### 3.1 Compositional Homomorphisms

We assume a fixed relational signature, with $R$ ranging over all relation symbols, and consider definitions of homomorphisms $\phi : A \to B$ of the form

$$\Delta[\phi] \quad \Leftrightarrow \quad l_1[\phi]; R^A; r_1[\phi] \quad \bowtie \quad l_2[\phi]; R^B; r_2[\phi] \tag{3.2}$$

where $l[\_]$'s and $r[\_]$'s are relational expressions (using only relational composition and inverse and parameterized by $\_$), and $\bowtie$ is one of the set-relations $\{=, \subseteq, \supseteq\}$.

**Definition 3.2** *A definition is* compositional *iff for all $\phi : A \to B$, $\psi : B \to C$, we have $\Delta[\phi]$ & $\Delta[\psi] \Rightarrow \Delta[\phi;\psi]$, i.e.:*

$$\begin{aligned} & l_1[\phi]; R^A; r_1[\phi] \bowtie l_2[\phi]; R^B; r_2[\phi] \quad \& \\ & l_1[\psi]; R^B; r_1[\psi] \bowtie l_2[\psi]; R^C; r_2[\psi] \\ & \qquad\qquad \Rightarrow l_1[\phi;\psi]; R^A; r_1[\phi;\psi] \quad \bowtie \quad l_2[\phi;\psi]; R^C; r_2[\phi;\psi] \end{aligned}$$

The number of syntactic expressions of the kind $l[\phi]$ is infinite, however, since homomorphisms are functions we have the simple fact:

**Fact 3.3** *a) $\phi^-;\phi;\phi^- = \phi^-$   b) $\phi;\phi^-;\phi = \phi$   c) $\phi^-;\phi = id_{\phi[\underline{A}]}$*

Thus the length of each of the expression $l[\phi]$, resp. $r[\phi]$ (measured by the number of occurring $\phi$'s or $\phi^-$'s) can be limited to 2.

On the other hand, both sides of a definition from (3.2) must yield relational expressions of the same type, i.e., of one of the four types $A \times A, A \times B, ...$, which will be abbreviated as $AA, AB, ...$

For each choice of $\bowtie$, this leaves us with four possibilities for each type. For instance, for $AB$ we have the following four possibilities:

$$\begin{array}{ll} \top_{AB} : \phi;\phi^-;R^A;\phi \ \bowtie \ \phi;R^B;\phi^-;\phi & \quad \bot_{AB} : R^A;\phi \ \bowtie \ \phi;R^B \\ E_{AB} : \phi;\phi^-;R^A;\phi \ \bowtie \ \phi;R^B & \quad W_{AB} : R^A;\phi \ \bowtie \ \phi;R^B;\phi^-;\phi \end{array}$$

The symbols denoting the respective possibilities are chosen for the following reason. Relational composition preserves each of the relations $\bowtie$, i.e., given a particular choice of $\bowtie$ and any relations $C, D$ (of appropriate type), we have: $R_1 \bowtie R_2 \Rightarrow C;R_1 \bowtie C;R_2$ and $R_1 \bowtie R_2 \Rightarrow R_1;D \bowtie R_2;D$. Starting with $\bot_{AB}$ and pre-composing (on the "$E$ast") both sides of $\bowtie$ with $\phi;\phi^-;(\_)$, we obtain $E_{AB}$; post-composing (on the "$W$est") both sides of $\bowtie$ with $(\_);\phi^-;\phi$, we obtain $W_{AB}$. Dual compositions lead from there to $\top_{AB}$. Thus we have that $\bot_{AB} \Rightarrow E_{AB}, W_{AB} \Rightarrow \top_{AB}$ and the corresponding lattices are obtained for the other three types starting, respectively, with

$$\bot_{AA} : R^A \ \bowtie \ \phi;R^B;\phi^- \qquad \bot_{BB} : \phi^-;R^A;\phi \ \bowtie \ R^B \qquad \bot_{BA} : \phi^-;R^A \ \bowtie \ R^B;\phi^-$$

Figure 1 shows the four lattices for each type (the choice of $\bowtie$ is the same for all of them).

The additional equivalences (indicated with dotted arrows) are easily verified using the fact that composition preserves each of $\bowtie$ and Fact 3.3. Also all the top definitions are equivalent which follows by simple calculation.

These observations simplify the picture a bit, leading, for each choice of $\bowtie$, to the order of 9 possible definitions shown in figure 2.

Furthermore, choosing $\subseteq$ for $\bowtie$, the above ordering collapses.

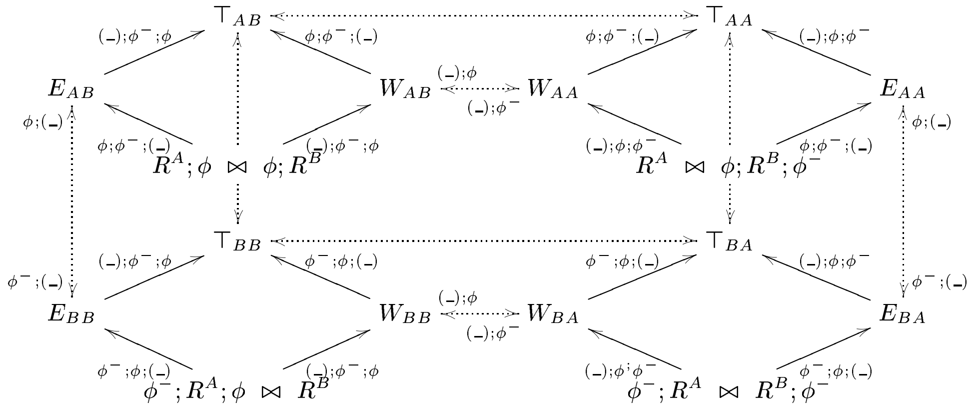



**Fig. 1.** Lattices for each relation type for each choice of $\bowtie$).

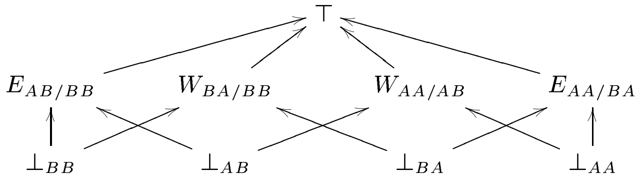



**Fig. 2.** Possible definitions (for a given choice of $\bowtie$).

**Proposition 3.4** *All definitions (of the form (3.2)) involving $\subseteq$ are equivalent.*

We are thus left with one definition involving $\subseteq$ and 18 other definitions obtained from two instances (with $=$, resp. $\supseteq$ for $\bowtie$) of the orderings in figure 2. The following, main theorem shows that only the bottom elements of these orderings yield compositional definitions.

**Theorem 3.5** *[30] A definition is compositional iff it is equivalent to one of the following forms:*

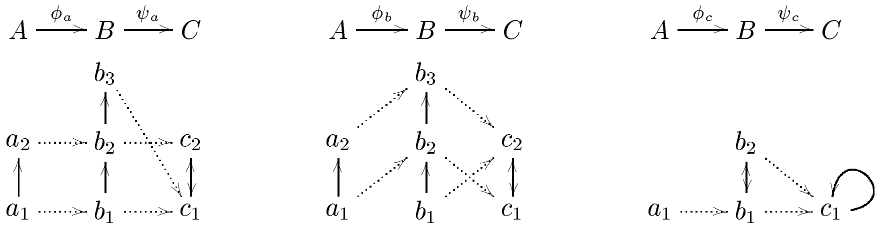
1) $R^A;\phi \bowtie \phi;R^B$ 2) $\phi^-;R^A;\phi \triangleright R^B$ 3) $\phi^-;R^A \triangleright R^B;\phi^-$ 4) $R^A \triangleright \phi;R^B;\phi^-$

*where* $\bowtie \in \{=,\subseteq,\supseteq\}$ *and* $\triangleright \in \{=,\supseteq\}$.

Proof: For the "if" part, one easily checks that 1)–4) do yield compositional definitions. In fact, this part of the theorem holds for *any* transitive set-relation $\bowtie$. For instance, for 3) we verify:

$$\begin{array}{rrcl}
 & \phi^-;R^A \bowtie R^B;\phi^- & \& & \psi^-;R^B \bowtie R^C;\psi^- \\
\Rightarrow & \psi^-;\phi^-;R^A \bowtie \psi^-;R^B;\phi^- & \& & \psi^-;R^B;\phi^- \bowtie R^C;\psi^-;\phi^- \\
 & & \Rightarrow & (\phi;\psi)^-;R^A \bowtie R^C;(\phi;\psi)^-
\end{array}$$

The "only if" part is shown providing counter-examples for the remaining possibilities. Although there are 10 cases left, they are easily shown by the following three counter-examples. In all cases, the given homomorphisms $\phi, \psi$ satisfy the respective definition with $=$ for $\triangleright$ (hence, also for $\supseteq$), while their composition does not satisfy the respective definition with $\supseteq$ for $\triangleright$. Thus we obtain immediately counter-examples for both $\triangleright \in \{=, \supseteq\}$.
Vertical arrows represent the relation $(R)$ in respective multialgebras; the dotted arrows illustrate the images under the respective homomorphisms:




**a**) **b**) **c**)

**a)** for $W_{BB}$ : $\phi^-; R^A; \phi \ \triangleright\ R^B; \phi^-; \phi$. We have: $\phi_a^-; R^A; \phi_a = R^B; \phi_a^-; \phi_a$ and $\psi_a^-; R^B; \psi_a = R^C; \psi_a^-; \psi_a$. However, for the composition $\rho_a = \phi_a; \psi_a$, we have $\langle c_2, c_1\rangle \in R^C; \rho_a^-; \rho_a$ but $\langle c_2, c_1\rangle \notin \rho_a^-; R^A; \rho_a$, i.e., $\rho_a^-; R^A; \rho_a \not\supseteq R^C; \rho_a^-; \rho_a$.
**b)** for $E_{BB}$ : $\phi^-; R^A; \phi \triangleright \phi^-; \phi; R^B$ is quite analogous. $\phi_b^-; R^A; \phi_b = R^B; \phi_b^-; \phi_b$ and $\psi_b^-; R^B; \psi_b = R^C; \psi_b^-; \psi_b$, but $\rho_b^-; R^A; \rho_b \not\supseteq \rho_b^-; \rho_b; R^C$ with $\langle c_2, c_1\rangle$ as a witness to this negation.
Both these examples can also be used as counter-examples for compositionality of $\top$, represented by $\top_{BB}$. For instance, in the first case, we have $R^B; \phi_a^-; \phi_a = \phi_a^-; \phi_a; R^B; \phi_a^-; \phi_a$ and the corresponding equality holds for $\psi_a$ and $R^C$ – so exactly the same argument yields a counter-example also for this case.
**c)** for both $W_{AA/AB}$ and $E_{AA/BA}$ (this is essentially example 3.1.b). Both $\phi_c$ and $\psi_c$ are obviously $W_{AB}$: $R^A; \phi_c = \phi_c; R^B; \phi_c^-; \phi_c$ and $R^B; \psi_c = \psi_c; R^C; \psi_c^-; \psi_c$. However, their composition yields: $\emptyset = R^A; \rho_c \not\supseteq \rho_c; R^C; \rho_c^-; \rho_c = \langle c_1, c_1\rangle$.
This gives also counter-example for $E_{BA}$ : $\phi_c^-; R^A \triangleright \phi_c^-; \phi_c; R^B; \phi_c^-$. □

This leaves us with 9 basic compositional definitions (more can be obtained by their conjunctions (see [11]). Inspecting the table 1, we can see that 1. and 2. define actually the same notion, and the only other compositional definitions are 9., 10. and 11.

Notice that, although we have used a rather special definition of relational composition, all counter-examples involve only binary relations. Thus, even if one defined composition of relations differently, as long as it subsumes the standard composition of binary relations, the theorem gives the maximal number of compositional definitions of homomorphisms.

On the other hand, one might probably come up with other forms of defining homomorphisms that are not covered by our theorem. [2] However, the majority (if not all) of commonly used forms do conform to this format. Occasionally, some authors consider certain modifications of the definitions from table 1. For instance, full outdegree and indegree homomorphisms (3,5,6) with the extra surjectivity requirement do compose. This restriction merely enforces the equality $\phi^-;\phi = id_B$, and leads to a special case of cases 2), 1) and 3) from the Theorem 3.5, respectively.

### 3.2 Congruences on multialgebras

Congruences of relational and power structures were studied, for instance, in [1–5]. As observed before, any equivalence gives rise to a (weak) homomorphism. However, the more specific definitions from theorem 3.5 may lead to more specific relations. We consider first equational definitions from the theorem, and characterize these kernels which turn out to be not merely equivalences but congruences of a sort. To proceed further we need a notion of a quotient:

**Definition 3.6** *Given a structure $A = \langle \underline{A}, R_1^A, R_2^A ...\rangle$ and an equivalence $\sim \subseteq \underline{A} \times \underline{A}$, a quotient $A/_\sim = Q$ is given by $\underline{Q} = \{[x] : x \in \underline{A}\}$ and $R_i^Q = \phi^-; R_i^A; \phi$, where $[x] = \{y \in \underline{A} : x \sim y\}$, and $\phi : \underline{A} \to \underline{Q}$ is defined by $\phi(x) = [x]$.*

Obviously, $\sim$ is the kernel of $\phi$, i.e., $x \sim y$ iff $\phi(x) = \phi(y)$, and $\sim = \phi;\phi^-$.

**Proposition 3.7** *Let $\sim$ be an equivalence on $A$ and $Q, \phi$ be as in def. 3.6.*

| | if $\sim$ satisfies | then |
|---|---|---|
| 2) | | $\phi^-; R^A; \phi = R^Q$ |
| 1) | $\sim; R^A; \sim = R^A; \sim$ | $R^A; \phi = \phi; R^Q$ |
| 3) | $\sim; R^A; \sim = \sim; R^A$ | $\phi^-; R^A = R^Q; \phi^-$ |
| 4) | $\sim; R^A; \sim = R^A$ | $R^A = \phi; R^Q; \phi^-$ |

In 1), 3) and 4) the relation $\sim$ is not just an arbitrary equivalence but has a flavour of a congruence:

1) can be stated as: $\forall \overline{a}, b : (\exists b' \sim b, \overline{a}' \sim \overline{a} : R^A(\overline{a}', b')) \Leftrightarrow (\exists b' \sim b : R^A(\overline{a}, b'))$, which gives: $\forall \overline{a}, b, \overline{a}' : R^A(\overline{a}, b) \wedge \overline{a} \sim \overline{a}' \Rightarrow \exists b' \sim b : R^A(\overline{a}', b')$;
3) yields a dual condition: $\forall \overline{a}, b, b' : R^A(\overline{a}, b) \wedge b' \sim b \Rightarrow \exists \overline{a}' \sim \overline{a} : R^A(\overline{a}', b')$;
4) is strongest: $\forall \overline{a}, b, \overline{a}', b' : R^A(\overline{a}, b) \wedge \overline{a}' \sim \overline{a} \wedge b' \sim b \Rightarrow R^A(\overline{a}', b')$.[3]

For any (at least weak) homomorphism we have the converse of 3.7:

[2] E.g., using complementation in addition to composition and inverse, or else using relations instead of functions as homomorphisms. We have to leave such generalizations to future research.

[3] Following a suggestion from an anonymous referee, one might want to compare this case to the notion of bisimulation of process.

**Proposition 3.8** *Given a homomorphism $\phi : A \to B$, let $\sim$ be the kernel of $\phi$*

| | if $\phi : A \to B$ satisfies | then $\sim$ is an equivalence and |
|---|---|---|
| 1) | $R^A; \phi = \phi; R^B$ | $\sim; R^A; \sim = R^A; \sim$ |
| 3) | $\phi^-; R^A = R^B; \phi^-$ | $\sim; R^A; \sim = \sim; R^A$ |
| 4) | $R^A = \phi; R^B; \phi^-$ | $\sim; R^A; \sim = R^A$ |

There is no line for condition 2) since $\phi^-; R^A; \phi = R^B$ obviously implies that $\sim$ is an equivalence but, in fact, this follows for any mapping $\phi$.

This is not the strongest formulation of this fact. For $\sim$ being an equivalence it suffices, of course, that $\phi$ is a weak homomorphism. Furthermore, for instance 1) implies $R^A; \phi = \phi; R^B; \phi^-; \phi$ which is sufficient to establish the respective property of $\sim$. In general, since $\sim$ is induced only from the image of $A$ under $\phi$, restricting the homomorphisms' definitions on the $R^B$-side to this image (i.e. by $\phi^-; \phi$) will yield the same properties of $\sim$.

Similar results do not follow for the "closed" versions, i.e., for the homomorphisms defined by $\supseteq$ in place of $=$. We can uniformly replace $=$ by $\supseteq$ in proposition 3.8, but then the statements in the right column are trivial for any mapping $\phi$. If the target algebra is total then the kernel may retain the flavour of congruence. But, in general, "closed" homomorphisms induce only equivalence.

## 4 Categories of Multialgebras

Theorem 3.5 gave three possibilities for $\bowtie$. The category substituting for $\bowtie$ the relation $\subseteq$ is called "weak", those with $=$ are "tight", and those with $\supseteq$ "closed". (These names are, to some extent, motivated by the tradition within partial algebras and multialgebras.) In lack of better names, we then call the categories of kind 1) "inner" (since $\phi^-, \phi$ occur "inside", closest to $\bowtie$), of kind 2) "left", of kind 3) "outer" and of kind 4) "right". For a given signature $\Sigma$, we thus have 9 categories, with homomorphisms $\phi : A \to B$ given by the respective compositional definition:

| | inner | left | outer | right |
|---|---|---|---|---|
| closed | $\mathsf{MAlg}_{IC}(\Sigma)$ : $R^A; \phi \supseteq \phi; R^B$ | $\mathsf{MAlg}_{LC}(\Sigma)$ : $\phi^-; R^A; \phi \supseteq R^B$ | $\mathsf{MAlg}_{OC}(\Sigma)$ : $\phi^-; R^A \supseteq R^B; \phi^-$ | $\mathsf{MAlg}_{RC}(\Sigma)$ : $R^A \supseteq \phi; R^B; \phi^-$ |
| tight | $\mathsf{MAlg}_{IT}(\Sigma)$ : $R^A; \phi = \phi; R^B$ | $\mathsf{MAlg}_{LT}(\Sigma)$ : $\phi^-; R^A; \phi = R^B$ | $\mathsf{MAlg}_{OT}(\Sigma)$ : $\phi^-; R^A = R^B; \phi^-$ | $\mathsf{MAlg}_{RT}(\Sigma)$ : $R^A = \phi; R^B; \phi^-$ |
| weak | $\mathsf{MAlg}_{W}(\Sigma) : R^A; \phi \subseteq \phi; R^B$ | | | |

Table 2 summarizes finite (co-)completeness of the respective categories: '+' means the existence of the respective (co-)limit for arbitrary $\Sigma$; '$-$' indicates that there exists a counter-example: a signature $\Sigma$ and $\Sigma$-multilagebras for which the respective (co-)limit does not exist.

With a few exceptions, the positive results are obtained by constructions similar to (though never the same as) those in the standard category of algebras. The proofs for the inner (and weak) categories can be found in [29], and for the remaining ones in [11].

| | initial obj. | co-products | co-equalizers | terminal obj. | products | equalizers |
|---|---|---|---|---|---|---|
| $\mathsf{MAlg}_W(\Sigma)$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| $\mathsf{MAlg}_{IC}(\Sigma)$ | $-$ | $-$ | $-$ | $+$ | $-$ | $-$ |
| $\mathsf{MAlg}_{IT}(\Sigma)$ | $-$ | $-$ | $+$ | $-$ | $-$ | $-$ |
| $\mathsf{MAlg}_{LC}(\Sigma)$ | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ |
| $\mathsf{MAlg}_{LT}(\Sigma)$ | $-$ | $-$ | $+$ | $-$ | $-$ | $-$ |
| $\mathsf{MAlg}_{OC}(\Sigma)$ | $+$ | $+$ | $-$ | $+$ | $-$ | $+$ |
| $\mathsf{MAlg}_{OT}(\Sigma)$ | $+$ | $+$ | $+$ | $(+)$ | ? | $+$ |
| $\mathsf{MAlg}_{RC}(\Sigma)$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| $\mathsf{MAlg}_{RT}(\Sigma)$ | $+$ | $-$ | $-$ | $-$ | $-$ | $+$ |

**Table 2.** Finite limits and co-limits in the categories of multialgebras

It has perhaps been a prevailing opinion among mathematicians interested in the question that weak homomorphisms of relational structures provide the most useful notion. They were certainly the most commonly used ones. The above results do, if nothing else, justify and demonstrate this opinion: the category $\mathsf{MAlg}_W(\Sigma)$ possesses many desirable properties not possessed by the other categories. There is, however, an exception to this claim, namely, the category $\mathsf{MAlg}_{RC}(\Sigma)$ which, too, is finitely complete and co-complete. A possible exception is also the category $\mathsf{MAlg}_{OT}(\Sigma)$. We are almost certain that terminal objects, marked with $(+)$, exist and can be obtained by an interesting kind of term-model construction which has been shown to work in special cases. Also products, marked with '?', remain still under investigation.[4] If the conjecture about completeness of $\mathsf{MAlg}_{OT}(\Sigma)$ turns out to be true, this might be the most interesting of all the investiagted categories. This can be further strengthened by observing that $\phi^-; R^A = R^B; \phi^-$ implies $R^A; \phi \subseteq \phi; R^B$, so that properties of the most common, weak homomorphisms are actually implied by outer-tightness.

## 5 Conclusions

We started by considering the relational structures but the suggested definition of composition of relations of arbitrary arities turned such structures into algebras – namely, multialgebras. These provide a convenient way for algebraic study of relational structures, as well as for modeling phenomena like partiality and nondeterminism within a unified framework.

We have given, in theorem 3.5, a general characterization of compositional homorphisms of multilagebras. The characterization applies equally to relational structures. Thus, from the manifold of alternative proposed definitions of such homomorphisms, we identified 9 which allow a categorical approach. We have also described the equivalence relations which emerge as kernels of the corresponding (compositional) homomorphisms and which have various flavours of congruences.

[4] Hopefully, the details and proofs of these constructions will be completed by the time of the conference. We suggest the interested reader to contact the first author.

We have then studied in detail the obtained categories with respect to the existence of (finite) limits and co-limits. The categories with weak and right-closed homomorphisms, $\mathsf{MAlg}_W(\Sigma)$, $\mathsf{MAlg}_{RC}(\Sigma)$, possess all constructions. The former has been widely used and is probably the most accepted standard. It remains to be seen whether the structural properties of the latter category demonstrated here will offer grounds for comparable applications. We conjecture that also the category $\mathsf{MAlg}_{OT}(\Sigma)$ is finitely (co-)complete and may be, in fact, highly interesting, but its terminal objects and products need further investigation.

The remaining categories have much poorer structural properties. Thus, accepting compositionality and the categorical properties we have studied as the criteria for the evaluation of the homomorphisms, our results leave quite limited choice. These criteria may, of course, be debatable and we by no means claim their universal validity. However, even if one does not want to base one's choice exclusively on these criteria, the results of this paper, in particular, the characterization of compositional definitions, can provide a useful tool preventing one from looking for new, idiosyncratic notions serving only very peculiar purposes.

We have studied only the categories of *all* multialgebras over a given signature. The really interesting question might be whether the properties of (co-)completeness can be lifted to axiomatic classes. The problem with answering such a question is that one would first have to decide on the logical language in which one writes the axioms. This question is far from settled and faces different proposals from different authors working on multialgebras. Still, even though we have not considered axiomatic classes, our results can serve, at least, the negative purpose. We have studied axiomatic classes over *empty* sets of axioms – hence, at least the negative results, the demonstrated non-existence of some constructions, will remain valid when lifted to axiomatic classes specified in arbitrary languages.

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