Combining specification formalisms in the 'general logic' of multialgebras

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Abstract. We recall basic facts about the institution of multialgebras, $\mathcal{MA}$, and introduce a new, quantifier-free reasoning system for deriving consequences of multialgebraic specifications. We then show how $\mathcal{MA}$ can be used for combining specifications developed in other algebraic frameworks. We spell out the definitions of embeddings of institution of partial algebras, $\mathcal{PA}$, and membership algebras, $\mathcal{MEMB}$ into $\mathcal{MA}$. We also show an alternative relation, namely, institution transformation of $\mathcal{PA}$ into $\mathcal{MA}$ and discuss its role as compared to the embedding.

1 Introduction

Multialgebras provide a powerful algebraic framework for specification – primarily, but not exclusively, of nondeterministic behavior [5, 21, 20]. A nondeterministic operation returns the set of all possible outcomes, so it is interpreted as a function from the carrier to the powerset of the carrier. We follow, and in few places generalize, the definitions of multialgebras from earlier works like [20, 19]. We summarize earlier results on multialgebras in section 2 leading to the fact that they form an exact institution $\mathcal{MA}$, [15]. This result underlies the study of parameterized specifications and specification of parameterized data types, reported in [8, 9] – it will be used as the current paper addresses the related issue of composing specifications. Section 2 ends with the presentation of a new, quantifier-free, sound and strongly complete logic for multialgebraic specifications. We thus obtain a general logic of multialgebras (in the sense of [11]). The central focus of the paper, however, is neither nondeterminism nor reasoning but, instead, the possibilities offered by $\mathcal{MA}$ to combine different algebraic specification formalisms in one framework (where the reasoning system can, of course, faciliate proving consequences of the combined specifications).

Section 3 presents embeddings of institutions $\mathcal{PA}$ of partial algebras and $\mathcal{MEMB}$ of membership algebras into $\mathcal{MA}$. We do it by means of an example in which a $\mathcal{PA}$-specification **Set** and a $\mathcal{MEMB}$-specification **Nat** are embedded into $\mathcal{MA}$, the former is augmented with nondeterministic choice $\sqcup$, and then extended to a parameterized specification $\sqcup$**Set**[**El**]; finally, the imported **Nat** specification is passed as an actual parameter. We thus illustrate the potential of combining specifications from various algebraic formalisms in a unified framework. We also indicate in the concluding section 4 the possibility of extending the model class of $\mathcal{PA}$ (and $\mathcal{MEMB}$) specifications with nondeterminism using institution transformations, [10], and refer to related work [7, 6] on how this can be utilized for a smooth introduction of flexible error recovery strategies into $\mathcal{PA}$-specifications.

For the reason of space limitations we have to assume the reader to be familiar not only with the general background on algebraic specifications and category theory, but also with the institutions and their mappings. We use the definitions and notation from [11].

2 The institution of multialgebras

We summarize the relevant notions about multialgebras (see e.g., [21]; [6] contains the proofs concerning the institution of multialgebras). The algebraic signature, $\Sigma = (\mathbf{S}, \Omega)$, and terms over Σ with variables from a set X, $\mathcal{T}_{\Sigma,X}$, are defined in the usual way. A multialgebra for a signature Σ is an algebra where operations may be set-valued.

Definition 1. *A multialgebra A for $\Sigma = (\mathbf{S}, \Omega)$ is given by:*

- *a set s^A, the carrier set, for each sort symbol $s \in \mathbf{S}$*
- *a subset $c^A \in \mathcal{P}(s^A)$, for each constant, $c :\to s$*
- *an operation $\omega^A : {s_1}^A \times \cdots \times {s_k}^A \to \mathcal{P}(s^A)$ for each symbol $\omega : s_1 \times \cdots \times s_k \to s \in \Omega$, where $\mathcal{P}(s^A)$ denotes the power set of s^A. Composition of operations is defined by pointwise extension, i.e., $f^A(g^A(x)) = \bigcup_{y \in g^A(x)} f^A(y)$.*

The disjoint union of the carrier set(s) of a multialgebra A is denoted by $|A|$. One sometimes demands that constants and operations are total [21, 19], i.e. never return the empty set – we will not make this assumption. An operation is total if it returns nonempty result set for all arguments (a partial operation returns empty result set for some arguments). An operation returning not more than one value for any argument is deterministic (a nondeterministic operation returns more than one value for some arguments). So an operation that is total and deterministic is a function.

We generalize earlier work by allowing not only the result sets of operations but also the carrier sets to be empty. Note that for a constant $c \in \Omega$, c^A denotes a (sub)set of the carrier s^A. Thus constants can be used for unary predicates (as will be done in 3.3, when we relate membership algebras with multialgebras).

We use weak homomorphisms of multialgebras (see [18] for alternatives).

Definition 2. *Given two Σ-multialgebras A and B, a (weak) homomorphism $h : A \to B$ is a set of functions $h_s : s^A \to s^B$ for each sort $s \in \mathbf{S}$, such that:*

- *$h_s(c^A) \subseteq c^B$, for each constant $c :\to s$*
- *$h_s(\omega^A(a_1, \ldots, a_n)) \subseteq \omega^B(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$, for each operation $\omega : s_1 \times \cdots \times s_n \to s \in \Omega$ and for all $a_i \in s_i^A$.*

The two definitions give us, for a signature Σ, the category of Σ-multialgebras, $\mathbf{MAlg}_\Sigma$, with Σ-multialgebras as objects and weak homomorphisms as arrows.

Multialgebraic specifications are written using the atomic predicates of "deterministic" equality and set inclusion. Since even atomic multialgebraic theories do not, in general, possess initial models, there seems to be little reason to restrict the formulae to Horn clauses and so we allow general sequents.

Definition 3. *Given a signature Σ and a set of variables X, the set $\mathcal{F}_{\Sigma,X}$ of well formed formulae is given by:*

1. *Atoms: if $t, t' \in \mathcal{T}_{\Sigma,X}$ then:*
   - *$t \doteq t' \in \mathcal{F}_{\Sigma,X}$ (equality) – t and t' denote the same one-element set.*
   - *$t \prec t' \in \mathcal{F}_{\Sigma,X}$ (inclusion) – the set t is included in the set t'.*
2. *$\Gamma \to \Delta \in \mathcal{F}_{\Sigma,X}$, when Γ, Δ are finite (possibly empty) sets of atoms.*

Occasionally, we may write $\neg\phi$, where ϕ is an atom, for $\phi \to$. Notice that we do not (have to) attach the variable contexts to formulae – empty carriers are treated differently, mainly by a slight modification of the notion of assignment.

Definition 4. *Given a set of variables X and a multialgebra A, an assignment α is a function:*

$$\alpha : X \to |A| \uplus \{\emptyset\}, \textit{ where } \alpha(x_s) = \emptyset \iff s^A = \emptyset$$

The subscript at x_s indicates the sort of the variable, so that x_s is assigned $\emptyset$ iff the sort s^A is empty. Otherwise variables are assigned individual elements – not sets thereof! (For convenience, we identify 1-element sets and individual elements.) Unlike the standard definitions, our definition guarantees the existence of an assignment even if the carrier $|A|$ is empty. Then any term with variables from empty sort will be empty, since operations in multialgebra applied to the empty set yield the empty set.

In the standard way, an assignment α to A induces a unique interpretation of any term t in A, denoted $\alpha(t)$. Now, satisfaction of formulae in A is defined first relatively to an assignment, written $\langle A, \alpha\rangle$.

Definition 5. *The satisfaction relation $\models$ in a multialgebra A is defined by:*

1. *$\langle A, \alpha\rangle \models t \prec t'$ iff $\alpha(t) \subseteq \alpha(t')$;*
2. *$\langle A, \alpha\rangle \models t \doteq t'$ iff $\alpha(t) = \{e\} = \alpha(t')$, for some $e \in |A|$;*
3. *$\langle A, \alpha\rangle \models \Gamma \to \Delta$ iff $(\exists \delta \in \Delta : \langle A, \alpha\rangle \models \delta \vee \exists \gamma \in \Gamma : \langle A, \alpha\rangle \not\models \gamma)$;*
4. *$A \models \phi$ iff $\forall \alpha : Var(\phi) \to A : \langle A, \alpha\rangle \models \phi$, where $Var(\phi)$ denotes all variables in ϕ*

Putting all the definitions together, we obtain the institution $\mathcal{MA}$, [6]. It has the following property, which is used in defining the semantics of parameterized specifications:

Proposition 1. *The model functor $\mathsf{Mod}_{\mathcal{MA}} : \mathbf{Sign}^{op} \to \mathbf{Cat}$ is finitely continuous.*

Since **Sign** is (finitely) cocomplete, we obtain that $\mathcal{MA}$ is an exact institution, [15]. Consequently, it satisfies the amalgamation lemma (for its formulation and proof the reader is referred to [17], where exact institutions are called institutions with composable signatures).

2.1 The Gentzen proof system

We give a quantifier free Gentzen style proof system $\mathcal{G}$ for multialgebraic specifications. First a notational convention.

Remark 1. According to point 2 in Definition 5, an equality may hold only if the carrier (of the respective sort) is non-empty. Given an algebra A, we have that:

$$A \models x_s \doteq x_s \iff s^A \neq \emptyset \text{ and } A \models \neg(x_s \doteq x_s) \iff s^A = \emptyset$$

Also, if $s^A = \emptyset$, we have for any terms t_s, t'_s: $A \models t_s \prec t'_s$ and $A \not\models \neg(t_s \prec t'_s)$

We introduce logical symbols abbreviating the formulae stating that a carrier is empty or not.

Definition 6. *We define the symbols $\mathcal{E}_s \equiv \neg(x_s \doteq x_s)$, for any $x_s \in X_s$, and $\neg\mathcal{E}_s \equiv x_s \doteq x_s$, for any $x_s \in X_s$. By Remark 1, for any algebra A:*

$$A \models \mathcal{E}_s \iff s^A = \emptyset \textit{ and } A \models \neg\mathcal{E}_s \iff s^A \neq \emptyset$$

The axioms and rules of the system $\mathcal{G}$ are given below. $\mathcal{G}$ allows us to derive sequents from a set Θ of sequents. (We also allow $\mathcal{E}_s$ as additional ground atomic formulae.) For deriving only multialgebraic tautologies, the system $\mathcal{G}^-$ containing only the logical axioms, the replacement and the expansion rules will suffice. For strong completeness, however, we need also the additional specific cut rules originating from [16]. Notice that these are much more tractable than the general cut, since they give a specific prescription as to what cut-formulae can be used in a bottom-up proof construction. These rules allow us to dispense with the quantifiers when proving consequences of axiomatic theories. (In [2], proving ϕ from a specification Θ required translation of the whole into a first-order formula, corresponding to $\forall(\Theta) \to \phi$. Besides the translation work, it required the use of quantifiers and limited the result to finite Θ only.) On the other hand, Definition 4 of assignment and Remark 1 explain that quantifiers can be avoided also when dealing with empty carriers. Except for the presence of the atomic predicate $\doteq$, these are the most significant improvements with respect to the closely related system from [2]. For the details concerning the system below, the reader is referred to [6].

Axioms
$\Gamma \to x \prec x, \Delta \ : x \in X$ $\qquad$ $\Gamma, \gamma \to \gamma, \Delta$ $\qquad$ $\Gamma, \mathcal{E}_s \to t_s \prec t'_s, \Delta$

Replacement rules

$$\frac{\Gamma, x \prec t \to \Delta, x \prec t'}{\Gamma \to \Delta, t \prec t'}$$
$t \notin X$, and $x \in X$ is fresh

$$\frac{\Gamma \to \Delta, x \prec t \mid \Gamma, x \prec t' \to \Delta}{\Gamma, t \prec t' \to \Delta}$$
$t \notin X$ and $x \in X$ arbitrary

$$\frac{\Gamma \to \Delta, y \prec t \mid \Gamma \to \Delta, x \prec f(\ldots, y, \ldots)}{\Gamma \to \Delta, x \prec f(\ldots, t, \ldots)}$$
where $y \in X$ arbitrary and $t \notin X$

$$\frac{\Gamma, y \prec t, x \prec f(\ldots, y, \ldots) \to \Delta}{\Gamma, x \prec f(\ldots, t, \ldots) \to \Delta}$$
where $y \in X$ is fresh and $t \notin X$

$$\frac{\Gamma \to \Delta, t \doteq x \mid \Gamma \to \Delta, t' \doteq x}{\Gamma \to \Delta, t \doteq t'}$$
$t, t' \notin X$ and $x \in X$ arbitrary

$$\frac{\Gamma, t_s \doteq x_s, t'_s \doteq x_s \to \Delta}{\Gamma, t_s \doteq t'_s \to \Delta}$$
$t_s, t'_s \notin X$ and $x_s \in X$ is fresh

$$\frac{\Gamma \to \Delta, t_s \prec x_s \mid \Gamma \to \Delta, x_s \prec t_s \mid \Gamma, \mathcal{E}_s \to \Delta}{\Gamma \to \Delta, t_s \doteq x_s}$$
where $x_s \in X$ and $t_s \neq x_s$

$$\frac{\Gamma, t_s \prec x_s, x_s \prec t_s \to \Delta, \mathcal{E}_s}{\Gamma, t_s \doteq x_s \to \Delta}$$
where $x_s \in X$ and $t_s \neq x_s$

$$\frac{\Gamma \to \Delta, t_s \prec x_s \mid \Gamma \to \Delta, x_s \prec t_s \mid \Gamma, \mathcal{E}_s \to \Delta}{\Gamma \to \Delta, x_s \doteq t_s}$$
where $x_s \in X$ and $t_s \neq x_s$

$$\frac{\Gamma, t_s \prec x_s, x_s \prec t_s \to \Delta, \mathcal{E}_s}{\Gamma, x_s \doteq t_s \to \Delta}$$
where $x_s \in X$ and $t_s \neq x_s$

Expansion rules

$$\frac{\Gamma, y \prec f(\overline{z}) \to \Delta}{\Gamma, y \prec x, x \prec f(\overline{z}) \to \Delta}$$
(sound for arbitrary $x \in X$)

$$\frac{\Gamma, x \prec y \to \Delta}{\Gamma, y \prec x \to \Delta}$$

$$\frac{\Gamma, z \prec f(\ldots, y, \ldots) \to \Delta}{\Gamma, y \prec x, z \prec f(\ldots, x, \ldots) \to \Delta}$$
(sound for arbitrary $x \in X$)

$$\frac{\Gamma, y \prec z \to \Delta}{\Gamma, y \prec x, x \prec z \to \Delta}$$

$$\frac{\Gamma, \mathcal{E}_s, x_{s'} \prec f(\ldots, y_s, \ldots), \mathcal{E}_{s'} \to \Delta}{\Gamma, \mathcal{E}_s, x_{s'} \prec f(\ldots, y_s, \ldots) \to \Delta}$$

Specific cut rules (relative to the non-logical axioms Θ)

$$\frac{\Gamma \to \Delta, \gamma'_1 \;\mid \ldots \mid\; \Gamma \to \Delta, \gamma'_n \;\mid\; \Gamma, \delta'_1 \to \Delta \;\mid \ldots \mid\; \Gamma, \delta'_m \to \Delta}{\Gamma \to \Delta}$$
for each axiom $\gamma_1, \ldots, \gamma_n \to \delta_1, \ldots, \delta_m \in \Theta$, with arbitrary renaming $'$ of variables

Writing $\mathcal{G}^-$ for the above system $\mathcal{G}$ without the specific cut rules, we have:

Proposition 2. *The systems are sound and complete – for every sequent ϕ:*

- $\vdash_{\mathcal{G}^-} \phi \iff \models \phi$
- *For every set of axioms $\Theta : \Theta \vdash_{\mathcal{G}} \phi \iff \Theta \models \phi$.*

3 Other specification frameworks in $\mathcal{MA}$

In this section we show the embeddings of two institutions – $\mathcal{PA}$ of partial algebras and $\mathcal{MEMB}$ of membership algebras – into $\mathcal{MA}$. By means of an example, we illustrate the power of $\mathcal{MA}$ for reuse and combination of specifications written in other algebraic formalisms. We start by taking a $\mathcal{PA}$-specification of sets and embed it to $\mathcal{MA}$ in 3.1; in 3.2 we extend the resulting specification of sets with a nondeterministic choice operation; in 3.3 we embed a $\mathcal{MEMB}$-specification of natural numbers to $\mathcal{MA}$; in 3.4 we extract a (sub)specification of elements from the specification of sets with nondeterministic choice, obtaining a parameterized specification, and we use the result of the embedding of naturals from $\mathcal{MEMB}$ as an actual parameter. Thus, we have combined $\mathcal{PA}$ and $\mathcal{MEMB}$ specifications in $\mathcal{MA}$, moreover we have extended the specifications with nondeterminism. Outline of the following example:

$$\begin{tikzcd}
\mathbf{Set}_{\mathcal{PA}} \arrow[rr, hook, "institution", "embedding"'] & & \mathbf{Set}_{\mathcal{MA}} \arrow[d, "extension"'] & \\
& & \sqcup\mathbf{Set}_{\mathcal{MA}} \arrow[dr, squiggly, "parameter"', "introduction"] & \\
& \mathbf{El}_{\mathcal{MA}} \arrow[rr, "\iota"'] \arrow[d, "instantiation"', "\mu"] & & \sqcup\mathbf{Set}[\mathbf{El}]_{\mathcal{MA}} \arrow[d, "\mu'"] \\
\mathbf{Nat}_{\mathcal{MEMB}} \arrow[r, hook, "institution", "embedding"'] & \mathbf{Nat}_{\mathcal{MA}} \arrow[rr, "\iota'"] & & \sqcup\mathbf{Set}[\mathbf{Nat}]_{\mathcal{MA}}
\end{tikzcd}$$

3.5 shows another way of including $\mathcal{PA}$ in $\mathcal{MA}$ and compares it to the embedding.

3.1 Partial algebras and multialgebras

We recall the basics of the institution of partial algebras, $\mathcal{PA}$ (e.g., [3, 4, 13]).

Definition 7. *A partial algebra A for a signature $\Sigma = (\mathbf{S}, \Omega)$ is given by:*

- *A set s^A for each sort $s \in \mathbf{S}$*
- *A partial function $\omega^A : {s_1}^A \times \cdots \times {s_n}^A \to s^A$, for each $\omega : s_1 \times \cdots \times s_n \to s \in \Omega$*

Given two partial algebras A and B, a (weak) homomorphism $h : A \to B$ is a set of total functions $h_s : s^A \to s^B$, for each sort $s \in \mathbf{S}$ such that:

- *$h_s(\omega^A(x_1, \ldots, x_n)) = \omega^B(h_{s_1}(x_1), \ldots, h_{s_n}(x_n))$ for each $\omega : s_1 \times \cdots \times s_n \to s \in \Omega$ and arguments $x_1, \ldots, x_n$, whenever $\omega^A(x_1, \ldots, x_n)$ is defined.*

A weak homomorphism h can be equivalently described as an ordinary homomorphism such that for each operation $\omega \in \Omega : h(\mathbf{dom}(\omega^A)) \subseteq \mathbf{dom}(\omega^B)$, where **dom** identifies ω's definition domain in a given algebra A.

Definition 8. *Formulae are universally quantified Horn clauses over existential equations, $t \overset{e}{=} t'$. Satisfaction by a partial algebra A is defined as follows (let $\alpha : X \to |A|$ range over standard assignments (total functions)):*

- $\langle A, \alpha \rangle \models (X; t \overset{e}{=} t')$ *iff* $\alpha(t)$ *and* $\alpha(t')$ *are defined and* $\alpha(t) = \alpha(t')$
- $\langle A, \alpha \rangle \models (X; a_1, \ldots, a_n \to a)$ *iff* $\exists i : 1 \leq i \leq n : \langle A, \alpha \rangle \not\models a_i$ *or* $\langle A, \alpha \rangle \models a$
- $A \models (X; \phi)$ *iff* $\forall \alpha : X \to |A| : \langle A, \alpha \rangle \models (X; \phi)$

Embedding $\mathcal{PA}$ into $\mathcal{MA}$. To relate multialgebras with partial algebras we observe the similarities between the element equality $\doteq$ and the existential equality $\overset{e}{=}$. In a multialgebra, the equality does not hold if one side is interpreted by a set with cardinality greater than one or by the empty set. The existential equality does not hold in a partial algebra if one side of the equality sign is undefined. These similarities suggest the straightforward translation: replace $\overset{e}{=}$ by $\doteq$. Each x in the variable context X in $(X; \phi)$, will give rise to an extra condition $x \doteq x$ in the antecedent of the translation of ϕ.[1]

On the semantic side, any Σ-partial algebra can be trivially converted into a Σ-multialgebra by making all undefined operations return the empty set. Since operations in multialgebra are strict on the empty set, the implicit strictness assumption from partial algebras will be enforced automatically.

Definition 9. *The functor $\beta^- : \mathsf{Mod}_{\mathcal{PA}}(\Sigma) \to \mathsf{Mod}_{\mathcal{MA}}(\Sigma)$ maps a partial algebra A to a multialgebra in the following way:*

- $|\beta^-(A)| = |A|$
- *for all* $\overline{x} \in |\beta^-(A)|$ *and* $f \in \Omega$: $f(\overline{x})^{\beta^-(A)} = \begin{cases} \{f(\overline{x})^A\} & -\text{if } \overline{x} \in \mathbf{dom}(f^A) \\ \emptyset & -\text{otherwise} \end{cases}$

For a homomorphism: $h \in \mathsf{Mod}(\Psi(\Sigma, \Gamma))$, we define $\beta^-(h) = h$.
For a multialgebra M where all operations are either deterministic or return empty set, $\beta(M)$ denotes the corresponding partial algebra, i.e., $\beta^-(\beta(M)) = M$.

Saying that a multialgebra and a partial algebra are "essentially the same", we will mean that they are obtained from each other by means of $\beta(\_)$, resp. $\beta^-(\_)$.

The embedding of $\mathcal{PA}$ into $\mathcal{MA}$ is now obtained by augmenting the partial algebra specification with additional axioms forcing all operations to return either a unique element or the empty set. This is the underlying model in partial algebras which in the generalized context of $\mathcal{MA}$ need explicit axioms. For an operation $f(\overline{x})$, the axiom forcing it to be empty or deterministic is of the form

[1] For the use with the logic $\mathcal{G}$, we would instead add, to the consequent of the translated ϕ, the disjunct $\mathcal{E}_s$ for each sort s for which there is a variable $x_s \in X$. I.e., $(\{x_s\}, \Gamma \to \alpha)$ would become $\Gamma \to \alpha, \mathcal{E}_s$, rather than $x_s \doteq x_s, \Gamma \to \alpha$. But we retain this later notation to emphasize that all translated formulae remain Horn clauses.

$y \doteq y, y \prec f(\overline{x}) \rightarrow f(\overline{x}) \doteq f(\overline{x})$, where y is a fresh variable. The example below shows the embedding of the $\mathcal{PA}$-specification of sets (it is the constant $\{\}$ which can be partial in this example).

$$
\begin{array}{ll}
\mathbf{Set}_{\mathcal{PA}} = & \\
\mathbf{S} : Set, El & \\
\Omega : \{\} : & \rightarrow Set \\
\quad \circ : El \times Set & \rightarrow Set \\
\Theta : 1. & \{x, y, S\}; \\
& x \circ (y \circ S) \stackrel{e}{=} y \circ (x \circ S) \\
\quad 2. & \{x, S\}; \\
& x \circ (x \circ S) \stackrel{e}{=} x \circ S
\end{array}
$$

$$
\begin{array}{ll}
\mathbf{Set}_{\mathcal{MA}} = & \\
\mathbf{S} : Set, El & \\
\Omega : \{\} : & \rightarrow Set \\
\quad \circ : El \times Set & \rightarrow Set \\
\Theta : 1. & x \doteq x, y \doteq y, S \doteq S \\
& \rightarrow x \circ (y \circ S) \doteq y \circ (x \circ S) \\
\quad 2. & x \doteq x, S \doteq S \\
& \rightarrow x \circ (x \circ S) \doteq x \circ S \\
\quad 3. & y \doteq y, y \prec x \circ S \rightarrow x \circ S \doteq x \circ S \\
\quad 4. & y \doteq y, y \prec \{\} \rightarrow \{\} \doteq \{\}
\end{array}
$$

The following lemma shows that the axioms of the above form (e.g., 3., 4.) are sufficient – multialgebras satisfying such axioms are "essentially" partial algebras.

Lemma 1. *Let* $SP = (\mathbf{S}, \Omega, \Theta)$ *be a specification in* $\mathcal{MA}$ *such that, for each operation* $f : \overline{s} \rightarrow s \in \Omega : SP \models y \doteq y, y \prec f(\overline{x}) \rightarrow f(\overline{x}) \doteq f(\overline{x})$, *where* y *is distinct from all* $\overline{x}$*. Then in any* $M \in \mathsf{Mod}_{\mathcal{MA}}(SP)$ *we have that for all* $\overline{x} \in |M| : f(\overline{x})^M = \emptyset$ *or* $f(\overline{x})^M$ *is deterministic.*

Proof. Let $M \in \mathsf{Mod}_{\mathcal{MA}}(SP)$ and $\alpha : \{y, \overline{x}\} \rightarrow M$ be an assignment. Two cases:

1. $\langle M, \alpha \rangle \not\models y \doteq y$: then the carrier set of s is empty so $\alpha(f(\overline{x}))^M = \emptyset$.
2. $\langle M, \alpha \rangle \models y \doteq y$: if $\langle M, \alpha \rangle \not\models f(\overline{x}) \doteq f(\overline{x})$ then, since $\langle M, \alpha \rangle \models y \doteq y, y \prec f(\overline{x}) \rightarrow f(\overline{x}) \doteq f(\overline{x})$, we must have that $\langle M, \alpha \rangle \not\models y \prec f(\overline{x})$, i.e., $\alpha(f(\overline{x}))^M = \emptyset$.

($\alpha(f(\overline{x}))^M$ may be $\emptyset$ also when some of $\overline{x}$ range over empty sorts, but this case is covered by point 2. Thus we do not need additional conditions $x \doteq x$ for $x \in \overline{x}$.)

$\square$

The lemma and the above discussion lead to the main fact:

Proposition 3. *There is an embedding* (Ψ, α, β) *of institutions from* $\mathcal{PA}$ *to* $\mathcal{MA}$*; the embedding is a simple map.*

- *The functor* $\Psi : \mathbf{Sign}_{\mathcal{PA}} \rightarrow \mathbf{Th}_{0\mathcal{MA}}$ *is given by:* $\Psi(S, \Omega) = ((S, \Omega), \Theta_\Sigma)$, *where* $\Theta_\Sigma = \{y \doteq y, y \prec f(\overline{x}) \rightarrow f(\overline{x}) \doteq f(\overline{x}) : f \in \Omega\}$.
  For morphisms $\Psi(\mu_S, \mu_\Omega)$ *is the identity.*
- *The natural transformation* $\alpha : \mathsf{Sen}_{\mathcal{PA}} \rightarrow \Psi; \mathsf{Sen}_{\mathcal{MA}}$ *is given by:*
  - $\alpha(t \stackrel{e}{=} t') \equiv t \doteq t'$ – *auxiliary definition for atoms*
  - $\alpha(\{x_1 ... x_k\}; a_1 ... a_n \rightarrow a) \equiv x_1 \doteq x_1 ... x_k \doteq x_k, \alpha(a_1) ... \alpha(a_n) \rightarrow \alpha(a)$

  Ψ *is extended to a functor* $\Psi : \mathbf{Th}_{0\mathcal{PA}} \rightarrow \mathbf{Th}_{0\mathcal{MA}}$ *by:* $\Psi(\Sigma, \Theta) = (\Sigma, \Theta_\Sigma \cup \alpha_\Sigma(\Theta))$.

– *The components of the natural transformation* $\beta : \Psi^{op}; Mod_{\mathcal{MA}} \to Mod_{\mathcal{PA}}$ *are* β*'s from Definition 9, i.e.:*
  - $|\beta_\Sigma(M')| = |M'|$
  - $f(x_1,\ldots,x_n)^{\beta_\Sigma(M')} = \begin{cases} undefined \text{ if } f(x_1,\ldots,x_n)^{M'} = \emptyset \\ x \text{ such that } f(x_1,\ldots,x_n)^{M'} = \{x\} \text{ otherwise} \end{cases}$

  This is a well defined partial algebra by lemma 1. For a homomorphism: $h \in \mathsf{Mod}_{\mathcal{MA}}(\Psi(\Sigma,\Theta))$*, we define* $\beta_\Sigma(h) = h$.

We also have an immediate consequence of the above construction:

Proposition 4. *For a* $\mathcal{PA}$ *theory* (Σ,Θ)*, the functor* $\beta_{(\Sigma,\Theta)}$ *is an equivalence (in fact, an isomorphism) of categories* $\mathsf{Mod}_{\mathcal{MA}}(\Psi(\Sigma,\Theta))$ *and* $\mathsf{Mod}_{\mathcal{PA}}(\Sigma,\Theta)$.

Proof. The inverse functor $\beta^-_{(\Sigma,\Theta)}$ sends a partial algebra $P \in \mathsf{Mod}_{\mathcal{PA}}(\Sigma,\Theta)$ onto a multialgebra $M' \in \mathsf{Mod}_{\mathcal{MA}}(\Psi(\Sigma,\Theta))$ such that $\beta(M') = P$, i.e., it is β^- from Definition 9. One verifies easily the isomorphism condition. □

As Mossakowski showed in [13], $\mathcal{PA}$ allows to specify exactly the finitely locally presentable categories [1], i.e. we have identified the sub-institution of $\mathcal{MA}$ allowing to specify these classes of models. Given a partial algebra specification SP, we call $\Psi(SP)$ a multialgebra specification of *partial form* and denote this sub-institution $\mathcal{MAPA}$. As the finitely locally presentable categories have initial models, we have also obtained a sub-institution of $\mathcal{MA}$ admitting such models.

3.2 Extending specifications with nondeterminism

In some cases it may be desirable to extend a specification with nondeterministic operations. A strategy for doing this is to embed the specification into $\mathcal{MA}$ and then adding the nondeterministic operations.

The obtained specification **Set** is extended with nondeterministic choice $\sqcup$.

$\sqcup\mathbf{Set}_{\mathcal{MA}} =$
$\mathbf{S} : Set, El$
$\Omega : \{\} : \quad \to Set$
$\quad \circ : El \times Set \to Set$
$\quad \sqcup : \quad Set \to El$
$\Theta :$ 1. $x \doteq x, y \doteq y, S \doteq S \to x \circ (y \circ S) \doteq y \circ (x \circ S)$
2. $x \doteq x, S \doteq S \to x \circ (x \circ S) \doteq x \circ S$
3. $y \doteq y, y \prec x \circ S \to x \circ S \doteq x \circ S$
4. $y \doteq y, y \prec \{\} \to \{\} \doteq \{\}$
5. $z \prec \sqcup(x \circ S) \to z \doteq x, z \prec \sqcup(S)$

$El^A = \{a,b,c\}$
$Set^A = \mathcal{P}^{fin}(|El|^A)$
$\{\}^A = \{\emptyset\}$
$x \circ^A S = \{x\} \cup S$
$\sqcup^A S = S$

A possible model A, shown on the right, has total and deterministic operations $\{\}$ and $\circ$, while $\sqcup(\{\})$ is undefined and $\sqcup(S)$ for non-empty S is nondeterministic.[2]

[2] One should be wary of the possible confusion in this example: the elements of the sort of defined sets Set obtain in A the interpretation as the actual, semantic sets of elements, except that $\{\}^A$ is interpreted as a well defined element $\{\emptyset\}$. The latter must be distinguished from the $\emptyset$ returned by the undefined operation $\sqcup(\{\})^A$.

3.3 Membership algebras and multialgebras

We now repeat the steps made for $\mathcal{PA}$ for the institution $\mathcal{MEMB}$ of membership algebras, [12]. We will use notation corresponding to the rest of the paper, which slightly differs from that used in [12].

Definition 10. *A (membership) signature Σ is a quadruple $\Sigma = (\mathbf{S}, \Omega, \Pi, \pi)$, where $(\mathbf{S}, \Omega)$ is a standard signature, Π is a set of sub-sort predicate names, and π is a function $\pi : \Pi \to \mathbf{S}$.*

The function π labels sub-sort predicate symbols by sort symbols – its intention is to identify a predicate p with $\pi(p) = s$ as a sub-sort of sort s.

Definition 11. *A signature morphism between two membership signatures $\Sigma = (\mathbf{S}, \Omega, \Pi, \pi)$ and $\Sigma' = (\mathbf{S}', \Omega', \Pi', \pi')$ is a triple $\mu = (\mu_\Pi, \mu_\mathbf{S}, \mu_\Omega)$, where $\mu_\Pi : \Pi \to \Pi'$ and $\mu_\mathbf{S} : \mathbf{S} \to \mathbf{S}'$ are functions such that the following diagram*

$$\begin{tikzcd}
\Pi \arrow[r, "\mu_\Pi"] \arrow[d, "\pi"'] & \Pi' \arrow[d, "\pi'"] \\
\mathbf{S} \arrow[r, "\mu_\mathbf{S}"'] & \mathbf{S}'
\end{tikzcd}$$

commutes, and there exists an ω' for each $\omega \in \Omega$ such that: $\mu_\Omega(\omega : \overline{s} \to s) = \omega' : \mu_\mathbf{S}(\overline{s}) \to \mu_\mathbf{S}(s)$.

Definition 12. *A membership algebra for a signature $(\mathbf{S}, \Omega, \Pi, \pi)$ is given by:*

- *a many sorted $(\mathbf{S}, \Omega)$ algebra A, together with*
- *an assignment of a subset $p^A \subseteq \pi(p)^A$ for each predicate name $p \in \Pi$.*

A membership algebra homomorphism $h : A \to B$ is an ordinary homomorphism which, in addition, satisfies: $h_{\pi(p)}(p^A) \subseteq p^B$.

Definition 13. *The axioms used for specifying classes of membership algebras are universally quantified Horn clauses over atomic formulae:*

- *equations, $t = t'$,*
- *membership assertions of the form $t : p$, for $p \in \Pi$ and $t \in (\mathcal{T}_{\Sigma,X})_{\pi(p)}$.*

The assignments to variables are as usual, and the satisfaction is defined as follows:

Definition 14. *Let $\alpha : X \to |A|$ range over standard assignments.*

1. *$\langle A, \alpha\rangle \models (X; t = t')$ iff $\alpha(t) = \alpha(t')$*
2. *$\langle A, \alpha\rangle \models (X; t : p)$ iff $\alpha(t) \in p^A$*
3. *$\langle A, \alpha\rangle \models (X; a_1, \ldots, a_n \to a)$ iff $\exists a_i : \langle A, \alpha\rangle \not\models (X; a_i)$ or $\langle A, \alpha\rangle \models (X; a)$*
4. *$A \models (X; \varphi)$ iff $\forall \alpha : X \to |A| : \langle A, \alpha\rangle \models (X; \varphi)$*

It is shown in [12] that the above definitions yield a liberal institution of membership algebras $\mathcal{MEMB}$. Moreover there exists an embedding of institutions both ways between $\mathcal{MEMB}$ and the institution of many sorted Horn logic with predicates and equalities, i.e. these two institutions can be viewed as sub-logics of each other.

Embedding $\mathcal{MEMB}$ into $\mathcal{MA}$ is based on the fact that nondeterministic constants play the same role as unary predicates. Hence the membership relation $t : p$ is naturally translated as $t \prec p$. Making, in addition, all operations deterministic, one obtains the straightforward translation of $\mathcal{MEMB}$ specifications into $\mathcal{MA}$, as shown in the example below:

$\mathbf{Nat}_{\mathcal{MEMB}} =$	$\mathbf{Nat}_{\mathcal{MA}} =$
$\mathbf{S} : Nat$	$\mathbf{S} : Nat$
$\Pi : \pi(nat) = Nat$	$\Omega : \ nat : \ \to Nat$
$\quad \pi(pos) = Nat$	$\quad pos : \ \to Nat$
$\Omega : zero : \ \to Nat$	$\quad zero : \ \to Nat$
$\quad suc : Nat \to Nat$	$\quad suc : Nat \to Nat$
$\quad pred : Nat \to Nat$	$\quad pred : Nat \to Nat$
$\Theta : 1.\ \emptyset; \quad zero : nat$	$\Theta : 1. \quad zero \prec nat$
$2.\ \{x\};\ x : pos \to x : nat$	$2.\ x \doteq x, x \prec pos \to x \prec nat$
$3.\ \{x\};\ x : nat \to pred(suc(x)) = x$	$3.\ x \doteq x, x \prec nat \to pred(suc(x)) \doteq x$
$4.\ \{x\};\ x : pos \to pred(x) : nat$	$4.\ x \doteq x, x \prec pos \to pred(x) \prec nat$
$5.\ \{x\};\ x : nat \to suc(x) : pos$	$5.\ x \doteq x, x \prec nat \to suc(x) \prec pos$
	$6. \quad zero \doteq zero$
	$7. \quad suc(x) \doteq suc(x)$
	$8. \quad pred(x) \doteq pred(x)$

Note that axioms 2., 3. and 4. could be written equivalently in $\mathcal{MA}$ as $pos \prec nat$; $pred(pos) \prec nat$; and $suc(nat) \prec pos$, i.e., using the syntax (and the intended, if not the formal, meaning) of unified algebras [14].

Proposition 5. *There is an embedding (Φ, α, β) of institutions from $\mathcal{MEMB}$ to $\mathcal{MA}$; the embedding is a simple map of institutions.*

- *The functor $\Phi : \mathbf{Sign}_{\mathcal{MEMB}} \to \mathbf{Th}_{\mathcal{MA}}$ is given by:*
  $\Phi(\mathbf{S}, \Omega, \Pi)$ is the theory $(\mathbf{S}, \Omega \uplus \Pi', \Theta_\Sigma)$, where:
  - $\Pi' = \{p :\to \pi(p) : p \in \Pi\}$ *– a new constant of sort $\pi(p)$ for each $p \in \Pi$.*
  - $\Theta_\Sigma = \{\omega(\overline{x}) \doteq \omega(\overline{x}) : \omega \in \Omega\}$ *– determinacy axiom for each operation.*

  $\Phi(\mu_S, \mu_\Omega, \mu_\Pi)$ is the signature morphism $(\mu_S, \mu_\Omega \uplus \mu_\Pi)$
- *The natural transformation $\alpha : \mathsf{Sen}_{\mathcal{MEMB}} \to \Phi; \mathsf{Sen}_{\mathcal{MA}}$ is given by:*
  - $\alpha(t : c) \equiv t \prec c$, *for each atom $t : c$*
  - $\alpha(t = t') \equiv t \doteq t'$, *for each atom $t = t'$*
  - $\alpha(\{x_1 ... x_k\}; a_1 ... a_n \to a) \equiv x_1 \doteq x_1 ... x_k \doteq x_k, \alpha(a_1) ... \alpha(a_n) \to \alpha(a)$

  Φ is extended to $\Phi : \mathbf{Th}_{0\mathcal{MEMB}} \to \mathbf{Th}_{0\mathcal{MA}}$ by: $\Phi(\Sigma, \Theta) = \Phi(\Sigma) \cup \alpha_\Sigma(\Theta)$).
- *The natural transformation $\beta : \Phi^{op}; \mathsf{Mod}_{\mathcal{MA}} \to \mathsf{Mod}_{\mathcal{MEMB}}$ is essentially the identity on models and homomorphisms. For any $M' \in \mathsf{Mod}_{\mathcal{MA}}(\Phi(\Sigma, \Theta))$*
  - $|\beta(M')| = |M'|$
  - $f(x_1, \ldots, x_n)^{\beta(M')} = x$ *such that* $f(x_1, \ldots, x_n)^{M'} = \{x\}$
  - $p^{\beta(M')} = p^{M'}$

  For a homomorphism $h : M' \to B' \in \mathsf{Mod}_{\mathcal{MA}}(\Phi(\Sigma, \Theta))$, we let $\beta(h) = h$.

Proposition 6. *The functor $\beta_{(\Sigma,\Theta)}$ is an equivalence (in fact, an isomorphism) of categories* $\mathsf{Mod}_{\mathcal{MA}}(\Phi(\Sigma,\Theta))$ *and* $\mathsf{Mod}_{\mathcal{MA}}(\Sigma,\Theta)$ *for every* $\mathcal{MEMB}$ *theory* (Σ,Θ).

Proof. The inverse functor $\beta^{-1}_{(\Sigma,\Theta)}$ sends a membership algebra $M \in \mathsf{Mod}(\Sigma,\Theta)$ onto a multialgebra $M' \in \mathsf{Mod}(\Phi(\Sigma,\Theta))$ such that $\beta(M') = M$ (i.e., $|M'| = |M|$, $f^{M'}(\overline{x}) = \{f^M(\overline{x})\}$ and for $p \in \Pi : p^{M'} = p^M$.) One verifies easily the isomorphism condition. $\square$

3.4 Parameterized specifications in $\mathcal{MA}$

Parameterized specifications and, in particular, specifications of parameterized data types in $\mathcal{MA}$, based on a generalization of the traditional persistency requirement, has been described in [8] and is further studied in [9]. In [9] we also address the issue of refinement of specifications based on introduction of additional structure into the specified programs. In the current context, we only claim that extracting a parameter from a flat specification can be seen as a refinement. We thus view now the specification $\sqcup\mathbf{Set}_{\mathcal{MA}}$ from 3.2 as a parameterized specification

$$\iota : \mathbf{El}_{\mathcal{MA}} \to \sqcup\mathbf{Set}[\mathbf{El}]_{\mathcal{MA}} \tag{1}$$

where the parameter specification $\mathbf{El}_{\mathcal{MA}}$ is simply the sort El.

Let $\mu : \mathbf{El}_{\mathcal{MA}} \to \mathbf{Nat}_{\mathcal{MA}}$ be the obvious specification morphism. In the standard way, we obtain the result of this parameter instantiation, $\sqcup\mathbf{Set}[\mathbf{Nat}]_{\mathcal{MA}}$, as the pushout in the category $\mathbf{Th}_{\mathcal{MA}}$ of multialgebraic specifications:

$$\begin{tikzcd}
\mathbf{El}_{\mathcal{MA}} \arrow[r, "\iota"] \arrow[d, "\mu"'] & \sqcup\mathbf{Set}[\mathbf{El}]_{\mathcal{MA}} \arrow[d, "\mu'"] \\
\mathbf{Nat}_{\mathcal{MA}} \arrow[r, "\iota'"'] & \sqcup\mathbf{Set}[\mathbf{Nat}]_{\mathcal{MA}}
\end{tikzcd}$$

Proposition 1 ensures the existence of well-defined semantics for such parameter instantiation in $\mathcal{MA}$. For the parameterized specification (1), one can even find a persistent functor $\mathsf{F} : \mathsf{Mod}_{\mathcal{MA}}(\mathbf{El}) \to \mathsf{Mod}_{\mathcal{MA}}(\sqcup\mathbf{Set}[\mathbf{El}])$:

$El^{\mathsf{F}(A)} = El^A$	$x \circ^{\mathsf{F}(A)} S = \{x\} \cup S$
$Set^{\mathsf{F}(A)} = \mathcal{P}^{fin}(El^A)$	$\sqcup^{\mathsf{F}(A)} S = S$

The model in 3.2 was obtained by applying this F to the algebra with $|El| = \{a,b,c\}$. Using Proposition 1, the persistent functor for the instantiated specification $\iota' : \mathbf{Nat}_{\mathcal{MA}} \to \sqcup\mathbf{Set}[\mathbf{Nat}]_{\mathcal{MA}}$ can be obtained from F by amalgamation.

3.5 Institution transformation of $\mathcal{PA}$ into $\mathcal{MA}$

$\mathcal{MA}$ offers wider possibilities of reuse and combination than embeddings exemplified so far. Instead of embedding other institutions, we can transform them into $\mathcal{MA}$. The transformation (proposed in [10] as one of the primitive operations for constructing various mappings of institutions) amounts, roughly, to extending the model class – for $\mathcal{PA}$ it will be like the embedding from proposition 3 but *without* adding the determinacy axioms:

$$\begin{array}{ll}
\mathbf{Set}_{\mathcal{PA}} = & \mathbf{Set'}_{\mathcal{MA}} = \\
\mathbf{S} : Set, El & \mathbf{S} : Set, El \\
\Omega : \{\} : \qquad\qquad \to Set & \Omega : \{\} : \qquad\qquad \to Set \\
\qquad \circ : El \times Set \to Set & \qquad \circ : El \times Set \to Set \\
\Theta : 1.\ \ \{x, y, S\}; & \Theta : 1.\ \ x \doteq x, y \doteq y, S \doteq S \\
\qquad\quad x \circ (y \circ S) \stackrel{e}{=} y \circ (x \circ S) & \qquad\quad \to\ x \circ (y \circ S) \doteq y \circ (x \circ S) \\
\qquad 2.\ \ \{x, S\}; & \qquad 2.\ \ x \doteq x, S \doteq S \\
\qquad\quad x \circ (x \circ S) \stackrel{e}{=} x \circ S & \qquad\quad \to\ x \circ (x \circ S) \doteq x \circ S
\end{array}$$

Embedding into $\mathcal{MA}$ allows one to augment existing operations with new, possibly nondeterministic ones, as was done in 3.2. Transforming, on the other hand, allows to extend the model class so that some old operations may acquire nondeterministic behavior. The resulting specification $\mathbf{Set'}_{\mathcal{MA}}$ above has a larger model class than the transformed $\mathbf{Set}_{\mathcal{PA}}$ – not only multialgebras which are "essentially" partial algebras, where $\{\}$ is undefined or deterministic, but also ones where it is nondeterministic.

Proposition 7. *There is an institution transformation* (Ψ, α, β) *of* $\mathcal{PA}$ *to* $\mathcal{MA}$.

- *The functor* $\Psi : \mathbf{Sign}_{\mathcal{PA}} \to \mathbf{Sign}_{\mathcal{MA}}$ *is identity on objects and morphisms.*
- *The natural transformation* $\alpha : \mathsf{Sen}_{\mathcal{PA}} \to \Psi; \mathsf{Sen}_{\mathcal{MA}}$ *is as in Proposition 3:*
  - $\alpha(t \stackrel{e}{=} t') \equiv t \doteq t'$ *for each atom* $t \stackrel{e}{=} t'$, *and*
  - $\alpha(\{x_1, ..., x_k\}; a_1, ..., a_n \to a) \equiv x_1 \doteq x_1, ..., x_k \doteq x_k, \alpha(a_1), ..., \alpha(a_n) \to \alpha(a)$
- *The components of the natural transformation* $\beta^- : Mod_{\mathcal{PA}} \to \Psi^{op}; Mod_{\mathcal{MA}}$ *are* β^-*'s from Definition 9, i.e.:*
  - $|\beta^-_\Sigma(P)| = |P|$
  - $f(x_1, \ldots, x_n)^{\beta^-_\Sigma(P)} = \begin{cases} \emptyset \text{ if } (x_1, \ldots, x_n) \notin \mathbf{dom}(f^P) \\ \{f(x_1, \ldots, x_n)^P\} \text{ otherwise} \end{cases}$

  For a homomorphism $h \in \mathsf{Mod}_{\mathcal{PA}}$, *we let* $\beta^-_\Sigma(h) = h$.

In [7, 6] this transformation was used for a smooth introduction of error recovery into specifications built in $\mathcal{PA}$. Only operations which in the original $\mathcal{PA}$-specifications might have been partial can now become nondeterministic and this was used to model partiality – interpreting undefinedness as indeterminacy of the possible result. As the nondeterminism can be narrowed down during the development process, this provided simple means for introduction of error elements or other recovery strategies, including even exceptions. In this case, we would claim, $\mathcal{MA}$ offers means for a unification of the two extremes in specification of partiality: the abstractness of $\mathcal{PA}$, due to the assumption of strictness, needed in the initial stages of development and, on the other hand, the possibility for detailed error treatment offered by total algebra approaches and useful in the final stages of low-level design.

4 Conclusion

Although the parameterization mechanism applied in 3.4 is quite standard, there are two points to observe. For the first, it is used in the new context of $\mathcal{MA}$, for which the relevant properties, in particular exactness, were verified. More importantly, the example from section 3 showed how to reuse (by embedding or transformation) and combine specifications developed in *different* frameworks. One should emphasize that this reuse is based on a rather straightforward translation of the theories which, for the most, amounts to axiomatizing the additional properties of various frameworks in order to restrict the generality of the multialgebraic semantics to the appropriate subset of models. As multialgebras extend in a conservative fashion many traditional algebraic frameworks, there is no need of *coding* or other complicated representation of the imported theories. Partial and membership algebras were chosen as an example, and a trivial embedding can be given of total algebras, but we expect that most algebraic specification formalisms can be fitted – and that also means: combined – within $\mathcal{MA}$. Except embedding, we have also shown another way of including institutions into $\mathcal{MA}$, exemplified by transformation of $\mathcal{PA}$. With reference to earlier work [7, 6], we suggested that this particular transformation offers a powerful way of combining the advantages of partial algebra and total algebra approaches to partiality, providing additional means for specifying error recovery strategies like exceptions. Details of embeddings, as well as other ways of relating/including various formalisms in $\mathcal{MA}$ remain to be investigated.

The paper presented also a new reasoning system for multialgebras. As compared to earlier ones it combines various aspects: it allows empty results as well as empty carriers, it addresses both inclusion and deterministic equality predicates, it is strongly complete and yet quantifier free. Each of the earlier systems, e.g. [5, 2, 20, 19], was weaker than the present one at least at one of the above points. In [6] a related (and logically equivalent) Rasiowa-Sikorski style system is presented, which is amenable to implementation. With respect to the combination of various specifications, it might be worthwhile to implement such a system in order to experiment with the common logical space for deriving consequences of the combination of specifications originating from different frameworks.

Besides defining explicitly relations to other specification frameworks and developing methodologies for combination of different specifications, the problem which still remains open concerns finding the greatest possible (if such exists) liberal subinstitution of $\mathcal{MA}$ – preferably, in form of verifiable restrictions on the syntax of specifications.

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